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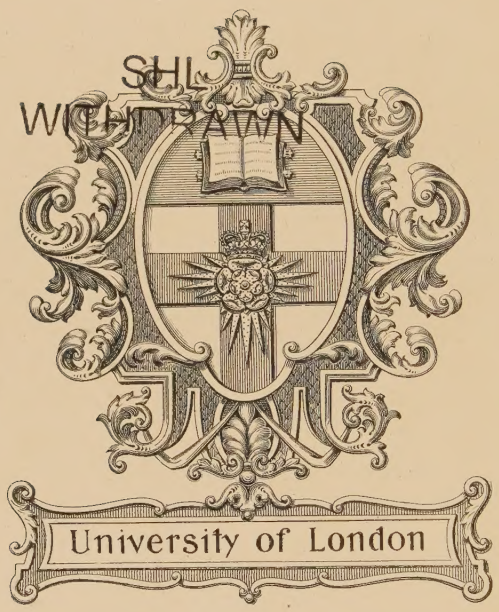

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PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY

SECOND SERIES

VOLUME 9

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1911

RECORDS OF PROCEEDINGS AT MEETINGS

SESSION NOVEMBER, 1910–JUNE, 1911.

Thursday, November 10th, 1910.

ANNUAL GENERAL MEETING.

Sir W. D. NIVEN, President, and, subsequently, Dr. H. F. BAKER, newly elected President, in the Chair.

Present nineteen members.

Miss M. Long and Prof. E. B. Wilson were elected members.

The Treasurer, Sir Joseph Larmor, presented his Report. On the motion of Lt.-Col. Cunningham, seconded by Mr. G. H. Hardy, the Report was received.

Dr. J. G. Leathem was reappointed Auditor.

Mr. Grace, as Secretary, reported that the number of members at the beginning of the current Session was 285. During the year three members had died, three had resigned, and five new members had been elected.

The Council and Officers for the ensuing Session were elected as follows:—President, Dr. H. F. Baker; Vice-Presidents, Mr. J. E. Cambbell, Major P. A. MacMahon, Sir W. D. Niven; Treasurer, Sir J. Larmor; Secretaries, Mr. J. H. Grace, Prof. A. E. H. Love; other members of the Council, Mr. G. T. Bennett, Dr. T. J. I'A. Bromwich, Prof. W. Burnside, Mr. E. Cunningham, Mr. A. L. Dixon, Dr. L. N. G. Filon, Prof. E. W. Hobson, Prof. H. M. Macdonald, Dr. A. E. Western.

Sir W. D. Niven delivered a Presidential Address on "The Relations of Mathematics and Experimental Science."

The following papers were communicated:—

*The Double Six: Mr. G. T. Bennett.

*The Existence of a Differential Coefficient: Dr. W. H. Young and Mrs. Young.

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* Printed in this volume.

(1) *Note on the Property of being a Differential Coefficient, (2) *Conditions that a Trigonometrical Series may have the Fourier Form: Dr. W. H. Young.

A Class of Integral Functions which includes Riemann's Zeta-function: Mr. F. Tavani.

The Geometrical Representation of Non-real Points in Space of Two and Three Dimensions: Mr. T. W. Chaundy.

†The Extension of Tauber's Theorem: Mr. J. E. Littlewood.

*The Stability of Rotating Shafts: Mr. F. B. Pidduck.

*A Class of Orthogonal Surfaces: Mr. J. E. Campbell.

*Non-integral Orders of Summability of Series and Integrals: Mr. S. Chapman.

Lineo-linear Transformations, specially in Two Variables: Dr. A. R. Forsyth.

Notes on Terminating Hypergeometric Series: Dr. W. F. Sheppard.

The Transformation of a Particular Type of Electromagnetic Field and its Physical Interpretation: Mr. H. Bateman.

Über die Dimensionentypen des Herrn Fréchet im Gebiete der linearen Mengen: Dr. P. Mahlo.

Thursday, December 8th, 1910.

Dr. H. F. BAKER, President, in the Chair.

Present thirteen members and two visitors.

Miss E. Coddington and Mr. T. C. Lewis were elected members.

Prof. Love presented the Report of the Auditor, Dr. J. G. Leathem.

On the motion of Lt.-Col. Cunningham, seconded by Mr. G. H. Hardy, it was agreed "that the Treasurer's Report be adopted, and that the thanks of the Society be voted to the Treasurer and the Auditor."

The following papers were communicated:—

(1) Properties of Logarithmico-exponential Functions, (2) Some Results concerning the Increase of Functions defined by an Algebraic Differential Equation of the First Order: Mr. G. H. Hardy.

Optical Geometry of Motion: Mr. A. A. Robb.

Note on the Pellian Equation: Mr. T. C. Lewis.

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* Printed in this volume.

† Printed in this volume with the title, "The Converse of Abel's Theorem on Power Series."

*On the Arithmetical Theory of Binary Cubic Forms: Mr. G. B. Mathews.

(1) *On the Integration of Fourier's Series, (2) *On the Theory of the Application of Expansions to Definite Integrals: Dr. W. H. Young.

Thursday, January 12th, 1911.

Dr. H. F. BAKER, President, in the Chair.

Present twenty members.

Messrs. S. Chapman, T. W. Chaundy, H. R. Hassé, and A. Lynch, were elected members.

Mr. T. C. Lewis was admitted into the Society.

The following papers were communicated:—

A Property of the Number Seven: Mr. T. C. Lewis.

On the Fundamental Theorem Relating to the Fourier Constants for Given Functions: Prof. E. W. Hobson.

The Integration of the Equations of Propagation of Electric Waves: Prof. H. M. Macdonald.

On the Fundamental Theorem in the Theory of Functions of a Complex Variable: Dr. W. H. Young.

On the 3–3 Birational Transformation in Three Dimensions (Second Paper): Miss H. P. Hudson.

Thursday, February 9th, 1911.

Dr. H. F. BAKER, President, in the Chair.

Present fifteen members.

The following papers were communicated:—

The Application of the Mathematical Theory of Relativity to the Electron Theory of Matter: Mr. E. Cunningham.

On the Reduction of Arithmetical Binary Forms which have a Negative Determinant: Messrs. G. B. Mathews and W. E. H. Berwick.

On Certain Vectors associated with an Electromagnetic Field and the Reflection of Light at the Surface of a Perfect Conductor: Mr. H. Bateman.

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* Printed in this volume.

Thursday, March 9th, 1911.

Dr. H. F. BAKER, President, in the Chair.

Present twenty-one members.

Prof. P. J. Heawood was elected a member.

The following paper was communicated:—

On the Reduction and Classification of Binary Cubic Forms which have a Negative Determinant: Mr. G. B. Mathews.

Informal communications were made as follows:—

On the Theory of Partitions: Major P. A. MacMahon.

On the Theory of the Transmission of Earthquake Waves: Prof. A. E. H. Love.

Thursday, April 27th, 1911.

Dr. H. F. BAKER, President, in the Chair.

Present fifteen members.

Mr. S. C. Basu was elected a member.

The following papers were communicated:—

Number of Primes of given Linear Forms: Lt.-Col. A. Cunningham.

The Properties of Certain Linear Homogeneous Substitutions: Mr. H. Hilton.

A Symmetrical Method of Generating Cubic Curves by Apolar Pencils: Mr. W. P. Milne.

On the Proofs of the Properties of Riemann's Surfaces discovered by Lüroth and Clebsch: Prof. M. J. M. Hill.

The Solution of the Homogeneous Linear Difference Equation of the Second Order (Second Paper): Mr. G. N. Watson.

A Cartesian Theory of Complex Geometrical Elements of Space: Mr. G. B. Mathews.

Thursday, May 11th, 1911.

Dr. H. F. BAKER, President, in the Chair.

Present seventeen members and a visitor.

Mr. J. H. Dibb was elected a member.

The following papers were communicated:—

The Kinematical and Geometrical Theory of a Deformable Octahedron: Mr. G. T. Bennett.

The Scattering of Light by a Large Conducting Sphere (Second Paper): Mr. J. W. Nicholson.

Thursday, June 8th, 1911.

Dr. H. F. BAKER, President, and, temporarily, Mr. J. E. CAMPBELL, Vice-President, in the Chair.

Present eleven members.

Mr. T. H. Gardner was elected a member.

Mr. S. Chapman was admitted into the Society.

The President announced that the Council had awarded the De Morgan Medal for 1911 to Prof. H. Lamb.

The following papers were communicated :—

On the Roots of Multiple Theta-Functions: Dr. H. F. Baker.

On the Multiplication of Dirichlet's Series: Mr. G. H. Hardy.

*On the Range of Borel's Method for the Summation of Oscillatory Series: Messrs. G. H. Hardy and J. E. Littlewood.

On the Convergence of Fourier Series and of the Allied Series: Dr. W. H. Young.

On some Two-Dimensional Problems in Electrostatics and Hydrodynamics: Mr. W. M. Page.

The Determination of all Groups of Rational Linear Substitutions of Finite Order which contain the Symmetric Group in the Variables: Prof. W. Burnside.

On the Nature of the Successions formed by the Coefficients of a Fourier Series: Dr. W. H. Young.

Informal communications were made as follows :—

*Note on Mersenne's Numbers: Lt.-Col. A. Cunningham.

On the Conditions that a Homogeneous Strain may be Reducible to a Plane Strain and an Extension at Right Angles to the Plane: Prof. A. E. H. Love and Dr. T. J. I'A. Bromwich.

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* See "Notes and Corrections" in this volume.

LIBRARY

Presents.

Between August, 1910, and May, 1911, the following presents were made to the Library, from their respective Authors or Publishers:—

Craig, J. S.—"Theory of Map-Projections, Egypt Survey Department," 1910.
Frank, Otto.—"Carl von Voit," 1910.
Kelvin, Lord.—"Mathematical and Physical Papers," vol. 5, 1911.
Legrand, Enrique.—"Sommations par une Formule d'Euler," 1911.
de Saussure, René.—"Exposé Résumé de la Géométrie des 'Feuillets,'" 1910.
Suini, A.—"La Confutazione della Geometria Non-Euclidea," 1911.
Timoschenko, S.—"Sur la Stabilité des Systèmes Élastiques," 1910.
———— "Einige Stabilitätsprobleme der Elastizitätstheorie," 1910.
———— "Erzwungene Schwingungen prismatische Stäbe," 1911.
Zeuthen, H. G. "Introduction à un traité didactique des méthodes énumératives de la Géométrie," 1909
"Indiana University Studies," vol. 8, no. 7.
"Kungliga Vetenskaps Societetens i Upsala Tv°hundraårsminne," 1910.
"Ole Rømers Adversaria," 1910.

Calcutta: Indian Engineering, vols. 47, 48, 1910; vol. 49, nos. 1–15, 1911.
Cape Town: South African Journal of Science, vol. 5, 1910; vol. 6, no. 1, 1911.
Coimbra: Accademia Polyt. de Porto, Ann. Scientificos, vol. 6, 1910; vol. 7, nos. 1–5, 1911.
Hamburg: Math. Gesellschaft Mittheilungen, bd. 5, heft 1, 1911.
Kansas: Univ. Science Bulletin, vol. 5, nos. 1–11, 1910.
London: Educational Times, vol. 63, nos. 591–596, 1910; vol. 64, nos. 597–601, 1911.
London: Educational Times, Math. Questions and Solutions, vols. 17, 18, 1910–11.
London: Mathematical Gazette, vol. 5, nos. 84–90, 1910; vol. 6, nos. 90, 91, 1911.
London: Nautical Almanac for 1913.
Madras: Indian Mathematical Club, Journal, vol. 2, 1910.
Paris: L'Enseignement Math., ann. 12, nos. 3–6, 1910; ann. 13, nos. 1, 2, 1911.
Tokyo: Sūgaku-Butsurigaku, vol. 5, nos. 13–22, 1910; vol. 6, nos. 1, 2, 1911.
Warsaw: Wiadomości Matem., tom. 14, 1910.

Exchanges.

Between August, 1910, and May, 1911, the following exchanges were received:—

American Journal of Mathematics, vol. 33, no. 2, 1911.
American Mathematical Society, Transactions, vol. 11, nos. 3, 4, 1910; vol. 12, nos. 1, 2, 1911.
American Mathematical Society, Bulletin, vol. 16, nos. 8–10; vol. 17, nos. 1–7; 1910–11.
American Philosophical Society, Proceedings, vol. 49, nos. 194–197, 1910.
Amsterdam: Nieuw Archief, deel 9, stuk 3, 1910.
Amsterdam: Revue Semestrielle, tome 18, 1910; tome 19, pt. 1, 1911.
Amsterdam: Wiskundige Opgaven, deel 10, stuk 5, 6, 1910.
Belgique: Académie Royale des Sciences, Bulletin, 1910; 1911, nos. 1, 2.

Berlin: Jahrbuch über die Fortschritte, bd. 39, hefte 1, 2, 1910.
Berlin: Journal für die Mathematik, bd. 37, 38, 1910; bd. 39, 1911.
Berlin: Sitzungsberichte der K. Preuss. Akademie, 1910, nos. 24–54.
Bologna: R. Accademia Memorie, ser. 6, tome 6, 1909.
Bordeaux: Société des Sciences, Mémoires, tome 5, cah. 1, 1910.
Bordeaux: Société des Sciences, Procès-Verbaux, 1908–9, 1909–10.
Bordeaux: Société des Sciences, Bulletin de la Commission Météorologique du Département de la Gironde, 1908, 1909.
Boston: American Academy of Arts and Sciences, vol. 45, nos. 3–21, 1910; vol. 46, nos. 1–17, 1911.
Cambridge Philosophical Society, Proceedings, vol. 15, pts. 5, 6, 1910; vol. 16, pt. 1, 1911.
Cambridge Philosophical Society, Transactions, vol. 21, pts. 12–14, 1911.
Cambridge, Mass.: Annals of Mathematics, vol. 11, no. 4, 1910; vol. 12, no. 1, 1911.
Canadian Institute, Transactions, no. 19, 1910.
Catania: Accademia Gioenia, Atti, ser. 5, vol. 3, 1910.
Catania: Accademia Gioenia, Bollettino, ser. 2, fasc. 11–15, 1910–11.
Dublin: Royal Irish Academy, Proceedings, ser. A, vol. 27, nos. 11, 12, 1909; vol. 28, no. 1, 1910.
Edinburgh: Mathematical Society, Proceedings, vol. 27, 1909.
Edinburgh: Royal Society, Proceedings, vol. 29, pts. 3–8, 1909; vol. 30, pts. 1–3, 1909.
Erlangen: Physik.-medicin. Societät, Sitzungsberichte, bd. 41, 1909.
France: Société Mathématique, Bulletin, tome 38, 1910; tome 39, fasc. 1, 1911.
Göttingen: Königl. Gesell. der Wissenschaften, Nachrichten, Math. Klasse, 1910; 1911, heft 1.
Göttingen: Königl. Gesell. der Wissenschaften, Nachrichten, Geschaft. Mitth., 1910.
Holland: Archives Néerlandaises, tome 15, 1910.
København: Kgl. Danske Vidensk Selsk. Skrifter, ser. 5, tome 1, no. 1; tome 8, nos. 1, 4; tome 9, no. 7; tome 10, nos. 4, 6, 9; tome 12, no. 5. Ser. 6, tome 1, no. 6; tome 2, nos. 6, 11; tome 3, no. 1; tome 5, no. 2; tome 7, no. 1; tome 8, no. 6; tome 10, no. 1. Ser. 7, tome 1, no. 6; tome 2, nos. 2, 5; tome 5, no. 1; tome 6, no. 1. 1849–1908.
København: Forhandlinger, Oversigt, 1910; 1911, no. 1.
Leipzig: Beiblätter zu den Annalen der Physik, bd. 34, hefte 9–24, 1910; bd. 35, hefte 1–8, 1911.
Leipzig: K. Sächsische Gesell., Math. Klasse, Berichte, 1909, nos. 4, 5; 1910, nos. 1–5.
Leipzig: K. Sächsische Gesell., Math. Klasse, Abhandlungen, bd. 32, no. 2, 1910.
Livorno: Periodico di Matematica, anno 25, fasc. 6, 1910; anno 26, fasc. 1–6, 1910–11.
Livorno: Periodico di Matematica, Supplemento, anno 13, fasc. 7–9, 1910; anno 14, fasc. 1–4, 1911.
Lombardo: Reale Istituto, Rendiconti, vol. 42, fasc. 16–20, 1909; vol. 43, fasc. 1–16, 1910.
Lombardo: Reale Istituto, Memorie, vol. 21, fasc. 1–4, 1910.
London: Institute of Actuaries, Journal, vol. 43, pt. 24, 1909; vol. 44, pt. 1, 1910.
London: Institution of Naval Architects, Transactions, vol. 50, 1910.
London: Nature, vols. 84, 85, and 86, nos. 2157–2166, 1910–11.
London: Physical Society, Proceedings, vol. 22, pts. 2–4, 1910; vol. 23, pts. 1–3, 1911.
London: Royal Society, Proceedings, Series A, vol. 83, no. 566, 1910; vol. 84, nos. 567–573, 575, 576, 1910-11.
London: Royal Society, Transactions, Series A, vol. 210, 1911.
Manchester Literary and Philosophical Society, Memoirs, vol. 53, pts. 2, 3, 1909; vol. 54, pt. 1, 1910.
Modena: Regia Accademia, Memorie, Ser. 3, vol. 8, 1909.
München: K. Bayer Akad. d. Wissen, Math.-Phys. Kl., Abhandlungen, bd. 25, abh. 1–4, 1909–10.
München: K. Bayer Akad. d Wissen. Math.-Phys. Kl., Sitzungsberichte, 1903–9; 1910, abh. 1–9.

Napoli: Accademia delle Scienze, Rendiconti, vol. 16, fasc. 1–12, 1910.
Napoli: Accademia delle Scienze, Atti, vol. 14, 1910.
Palermo: Rendiconti del Circolo Matematico, tome 30, 1910; tome 31, fasc. 1, 2, 1911.
Paris: Bulletin des Sciences Mathématiques, Mar.–Dec., 1910; Jan.–April, 1911.
Paris: Journal de l'Ecole Polyt., cah. 14, 1910.
Pisa: R. Scuola Normale Superiore, Annali, vol. 11, 1910.
Roma: Reale Accademia dei Lincei, Rendiconti, vol. 19, sem. 1, fasc. 9–12, sem. 2, 1910; vol. 20, sem. 1, fasc. 1, 1911.
Roma: Reale Accademia dei Lincei, Rendiconti delle Sedute Solenni, vol. 2, pp. 441–502, 1910.
Stockholm: Acta Mathematica, bd. 33, pts. 3, 4, 1910; bd. 34, pts. 1–3, 1911.
Torino: R. Accademia delle Scienze, Atti, vol. 15, disp. 11–15, 1910.
Toulouse: Faculté des Sciences, Annales, ser. 3, tome 1, fasc. 1, 2, 1909.
Wien: Monatshefte für Mathematik, jahr 22, viertel. 1, 1911.
Zurich: Vierteljahrsschrift, hefte 1, 2, 1910.

International Catalogue of Scientific Literature.

In the year April, 1909, to March, 1910 (inclusive), the following exchanges were sent in the first instance to Prof. Love to be indexed for the International Catalogue of Scientific Literature:—

"Proceedings of the Edinburgh Mathematical Society," Vol. XXVIII, 1910.
"Transactions of the Institution of Naval Architects," London, 1910.
"Journal of the Institute of Actuaries," Vol. XLIV, Pts. 2–4, and Vol. XLV, Pt. 1.
"Proceedings of the Manchester Literary and Philosophical Society," Vol. LIV, Pts. 2, 3, and Vol. LV, Pt. 1.
"Proceedings of the Royal Society of Edinburgh," Vol. XXX, Pts. 4–7, and Vol. XXXI, Pts. 1, 2.
"Proceedings of the Royal Irish Academy," Vol. XXVIII, Section A, Nos. 2, 3.

The following were also sent for the purposes of the Catalogue:—

"Mathematical Gazette," Nos. 84–91; London, 1910–11.
"Transactions of the Insurance and Actuarial Society of Glasgow," Ser. 6, Nos. 10, 11, 1909–10.
"Educational Times," Nos. 588–599; London, 1910–11.

OBITUARY NOTICE

ROBERT HARLEY

Robert Harley, though not an original member of our Society, was elected, on October 16th, 1865, in the first year of its existence. The first volume of our *Proceedings* contains a paper on "Differential Resolvents" from his pen. Only De Morgan's inaugural address, a syllabus by Sylvester, and a paper by Cayley, precede it in the volume.

He came of a Scotch family, but was born, on January 23rd, 1828, in Liverpool, where his father, who was a leading lay worker among the Wesleyan Methodists and afterwards became a minister of a seceding Methodist body, had settled. His school education was short and must have been somewhat erratic—partly, no doubt, because of the frequent changes of residence incidental to a Methodist minister's life, and also, perhaps, partly because of the wayward workings of a keen and masterful young intelligence. Only as the end of his school days approached, when at the age of fourteen he first learned the multiplication table, did he begin to find out mathematics. Then he let himself go. In a year he had ravenously devoured, and digested, all the mathematical works which he found accessible, and within two years we find him mathematical master "in a high-class school" near Liverpool. One year later, at seventeen, he was transferred as "Head Assistant" to the school at Blackburn in which his own efforts had been most greatly stimulated.

Here he worked early and late—especially early—at his absorbing mathematics, as well as at his teaching duties. There was no *Educational Times* or *Mathematical Gazette* then, but young and eager mathematicians, especially self-taught ones, had their annual feast of set and solved problems in the *Lady's and Gentleman's Diary*. This was his delight. May the old Annual never be forgotten! It brought Harley and (Sir James) Cockle together. In an indirect way, and at a later time, the resulting friendship in rivalry and collaboration led to the formation of another close tie between Harley and an older mathematician of the first rank—George Boole. This time Harley was disciple as well as friend.

See his essay on Boole's life and writings in the *British Quarterly Review* for July, 1866. The two friendships had much to do with the development of Robert Harley the mathematician.

But the pursuit of mathematics was only the recreation and the pride, not the business, of his life. Robert Harley, the mathematician, was, in fact, the Reverend Robert Harley seeking mental refreshment:

"rationibus agitandis exquirendisque, cum oblectatione sollertiae; qui est unus suavissimus pastus animorum."

At the age of twenty-three he entered a theological college (Airedale), at which he was later to be, for a short time, a professor, and after two years he was ordained a Congregationalist minister. Most of his mathematical writings were thought out and penned in the intervals of relaxation allowed by devoted attention to pastoral duties. Hence it arises that, though he was potentially a consummate investigator, his name will be associated with monographs which arrested attention and excited emulation rather than with any great scientific edifice of his own completion.

His published mathematical papers are about twenty-five in number. Our Society printed two of them; many are to be found in the *Memoirs* and *Proceedings of the Manchester Literary and Philosophical Society;* others in the *Quarterly Journal* and in the *British Association Reports;* and one in the *Philosophical Transactions of the Royal Society.* His productiveness was greatest during his first pastorate at Brighouse, in Yorkshire, which ended in 1868.

His earliest paper, "On Impossible Equations," is interesting from its title and from the indication which it affords of the bent of his mind. Very soon he became fascinated with the question of the insolubility by radicals of the general quintic equation. Probably, like Cockle, he began by distrusting the arguments by which this insolubility had been established. Be this so or not, he was led to a searching examination into what could be done by way of throwing light on the question. With sound intuition and much patient labour, he elaborated the theory of a sextic resolvent of the quintic. Of his numerous writings in this connexion, the two most important bore the title, "On the Theory of Symmetric Products," and appeared, one in the *Manchester Memoirs* for 1860, and the other in the *Philosophical Transactions* for 1861. The investigation, which was participated in by Cockle—the two were in almost daily correspondence—aroused the warm admiration of Cayley, who at first received it as new in essence as well as in execution, and brought his own great powers to bear on its continuation. Afterwards it was discovered that Jacobi earlier, and Malfatti even before him, had expounded the doctrine

NOTES AND CORRECTIONS

Dr. H. F. Baker's paper cited in the footnote on p. 177 is published in *Proc. R. Soc.*, London, Ser. A, Vol. 84, pp. 597–602.

Messrs. G. H. Hardy and J. E. Littlewood send the following abstract of their paper entitled "The Range of Borel's Method of Summation of Oscillatory Series," read at the June meeting :—

It appears from recent investigations (Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 8, p. 315; Littlewood, *ibid.*, Ser. 2, Vol. 9, p. 434) that a series Σa_n for which $|na_n| < K$ cannot be summable by any of Cesàro's means without being convergent; in fact, the existence of Abel's limit $\lim_{x\to 1} \Sigma a_n x^n$ is in such cases sufficient to ensure convergence. An analogous result holds for Riesz's more general methods. In this paper a similar theorem is proved relating to Borel's exponential method of summation : viz.,

(1) *If* Σa_n *is summable* (B), *and* $|\sqrt{n}\,.\,a_n| < K$, *then* Σa_n *is convergent.*

Further,

(2) *If* Σa_n *is summable* (B), *and* $|S_n| < K\sqrt{n}$, *then* Σa_n *is summable* $(C1)$.

These results may be applied to the problem of the multiplication of series. For example, it is easy to prove that

(3) *If* Σa_n, Σb_n *are summable* (B) *and* $|na_n| < K$, $|nb_n| < K$, *then the product series is summable* (B), and (1) shows that this is equivalent to the theorem (Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 6, p. 410) that *if the original series are convergent the product series is convergent.*

Further,

(4) *If* Σa_n *is absolutely summable* (B), *and* Σb_n *summable* (B), *and* $|n^r a_n| < K$, $|n^s b_n| < K$, *where* $r + s > \frac{3}{2}$, *then the product series is convergent.*

In particular, the first condition is satisfied if Σa_n is convergent to sum S and

$$|S_n - S| < Kn^{-\frac{1}{2}-\delta}.$$

Greater generality can doubtless be given to these results, which are mentioned only to show the kind of application which may be made of (1).

Lieut.-Col. A. Cunningham sends the following abstract of his communication made at the June meeting :—

Mr. H. J. Woodall, of Stockport, has recently found that $(2^{181}-1)$ is divisible by 43441. This leaves only 16 of the numbers (2^q-1), originally reported composite by Mersenne, still unverified.

Mr. G. T. Bennett sends the following correction of his paper on "The Composition of Finite Displacements ..." :—

p. 277, l. 9 from foot, *for* "lines a and d" *read* "lines b and d."

PAPERS

PUBLISHED IN THE

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

ON 8-VIC, 16-IC, &c. RESIDUACITY

By LT.-COL. ALLAN CUNNINGHAM, R.E.
(Fellow of King's College, London).

[Received January 6th, 1909.—Read January 14th, 1909.—Revised February 14th, 1910.]

[The author is indebted to Dr. A. E. Western for determining the theoretical correctness of the Criteria of Art. 11*b*, and for general revision of this paper.]

1. *Residuacity.*—By Fermat's theorem—

$$q^{p-1} \equiv +1 \pmod{p}, \quad [\text{when } q \text{ prime to } p], \textit{ always}, \tag{1}$$

and, writing—for shortness— $E = 2^{\kappa}$, (2)

it may happen that

$$q^{(p-1)/E} \equiv +1 \pmod{p} \quad [\text{if } p = E.\varpi + 1], \tag{3}$$

which is often written—for shortness—

$$(q/p)_E = +1, \tag{3a}$$

and, in this case, q is said to be a *residue* of the hyper-even order E, of the modulus p.

1*a*. *Scope of Paper.*—The object of this paper is to show that the residuacity of orders $E = 4, 8, 16$ can be determined for any base $(\pm q)$ whatever—(prime, of course, to p)—from *one*, *two*, or *three* of the simplest quadratic partitions of p; and to deduce the actual conditions themselves in a simple form for several small bases $q = 3, 5, 7, 11$.

1*b*. *Notation.*—In this paper all numbers are (positive) *integers :*

p denotes a prime ; q is a base prime to p.

ω, ϵ denote *odd* and *even* numbers respectively ; i an *integer*.

2. *Preliminary Conditions.*—In order that $(q/p)_E = \mp 1$, the following set of conditions are a necessary preliminary :—

Linear.— $$p = E.\varpi + 1. \tag{4}$$

Residuary.—$(q/p)_2 = +1$, $(q/p)_4 = +1$, $(q/p)_8 = +1$, ...,

ending with $(q/p)_{\frac{1}{2}E} = +1$. (5)

When these conditions are satisfied, then

$$(q/p)_E = \mp 1, \quad \textit{always}. \tag{6}$$

The further relation between q, p which serves to distinguish between the cases of the $\mp$ signs in (6) is called the *criterion* of residuacity (of order E) of the base q.

3. *Criteria for* $\pm q$.—Since

$$(q/p)_E.(\bar{q}/p)_E = (\bar{1}/p)_E = -1, \quad \text{if } p = E.\omega + 1, \tag{7a}$$

$$= +1, \quad \text{if } p = E.\epsilon + 1, \tag{7b}$$

the criterion for either of $\pm q$ can be immediately derived (at sight) from that of the other one, so that it suffices to investigate the criterion for only one of them : it is, however, often convenient to have both expressions available.

4. 2-*ic Congruences.*—Suppose the following 2-ic congruences to exist :—

$$q \equiv x_1^2, \quad x_1 \equiv x_2^2, \quad x_2 \equiv x_3^2, \quad \ldots, \quad x_{\kappa-1} \equiv x_\kappa^2 \pmod{p}. \tag{8}$$

These involve, and are involved in,

$$q \equiv x_1^2 \equiv x_2^4 \equiv x_3^8 \equiv x_4^{16} \equiv \ldots \equiv x_{\kappa-1}^{\frac{1}{2}E} \pmod{p}. \tag{8a}$$

Hence $(q/p)_4 = (x_1/p)_2$, $(q/p)_8 \equiv (x_2/p)_2$, $(q/p)_{16} = (x_3/p)_2$, ..., $(q/p)_E = (x_{\kappa-1}/p)_2 = \mp 1$. (8*b*)

The conditions (4) to (6) of residuacity of order E, and the above set (8) of 2-ic congruences, mutually involve one another. Thus the finding of the criterion of order $E = 2^\kappa$ is reducible to 2-ic residue work by aid of a system of only $(\kappa - 1)$ congruences : so that the use of one, two, or three congruences suffices to reduce the finding of the 4-tic, 8-vic, or

16-ic criteria respectively to 2-ic residue work. It is, however, often convenient to use 2-ic partitions of p instead of congruences: the same number of (independent) partitions should suffice (as a 2-ic partition is convertible into a congruence).

5. *2-ic Partitions of p.*—The five following 2-ic partitions of p will be used:—

$$(a^2+b^2),\quad (c^2+2d^2),\quad (e^2-2f^2),\quad \frac{1}{\lambda}(t^2+qu^2),\quad \frac{1}{\lambda'}(t'^2-qu'^2), \tag{9}$$

and will be styled—for shortness—

$$\text{The (a, b), (c, d), (e, f), } (t, u), (t', u') \text{ partitions,} \tag{9a}$$

It will be noted that the latter pair (t, u), (t', u') alone involve q.

5*a*. *Partitions when* $p-1 = 4\varpi$.—The preliminary conditions, E being here $\not< 4$, are (Art. 2)

$$p = 4\varpi+1 \quad \text{and} \quad (\bar{q}/p)_2 = (q/p)_2 = +1. \tag{4, 5}$$

These are known to involve

$$p = a^2+b^2 = \frac{1}{\lambda}(t^2+qu^2) = \frac{1}{\lambda'}(t'^2-qu'^2), \tag{10}$$

where λ, λ' are such that

$$\left.\begin{aligned} &\lambda = \lambda' = 1 \text{ in many* simple cases,}\\ &(\lambda/p)_2 = (\lambda'/p)_2 \text{ always}\\ &\lambda, \lambda' \text{ may be taken† squares, if required.}\end{aligned}\right\} \tag{10a}$$

The three partitions (10) are not independent, but form a *2-ic Triad*, the property of which is that any two of the partitions completely determine the third (when λ, λ' are known).

5*b*. *Partitions when* $p-1 = 8\varpi$.—The preliminary conditions of residuacity, with $E \not< 4$, are (Art. 2)

$$p = 8\varpi+1, \quad (\bar{q}/p)_2 = (q/p)_2 = +1. \tag{4, 5}$$

These are known to involve

$$p = a^2+b^2 = c^2+2d^2 = e^2-2f^2 = \frac{1}{\lambda}(t^2+qu^2) = \frac{1}{\lambda'}(t'^2-qu'^2), \tag{11}$$

wherein λ, λ' are restricted as in Art. 5*a*.

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* $\lambda = 1$, $\lambda' = 1$ when $\pm q$ is an *Idoneal* [*i.e.*, the determinant ($\mp q$) of a 2-ic form possessing only one class per genus], and also in many other cases.

† See Art. 3*c* of the author's paper "On 4-tic Residuacity, &c.," *Proc. London Math. Soc.*, Ser. 2, Vol. 1. The proof of this is due to Dr. A. E. Western.

The five partitions are not independent, but may be arranged in two 2-*ic Triads*

$$(a, b),\ (c, d),\ (e, f)\ ;\quad (a, b),\ (t, u),\ (t', u'), \tag{12}$$

the property of a Triad being that any two partitions thereof completely determine the third (when λ, λ' are known). Here the partition (a, b) is a member of both Triads : hence

$$(a, b) \text{ with one of } (c, d), (e, f) \text{ and one of } (t, u), (t', u') \tag{13}$$

suffice to completely determine the five partitions, so that only three are independent.

6. *Use of* (t, u), (t', u') *Forms.*—From these two forms it follows at once that

$$(\bar{q}/p)_E = (t/p)_{\frac{1}{2}E} \div (u/p)_{\frac{1}{2}E} \quad \text{and} \quad (q/p)_E = (t'/p)_{\frac{1}{2}E} \div (u'/p)_{\frac{1}{2}E}. \tag{14}$$

The evolution of the criterion of residuacity of order E is hereby *reduced to that of half* that order $(\frac{1}{2}E)$, and is presented in such a simple form that it can always be evolved when the criteria of the lower orders have been themselves completely worked out. This is the advantage of the use of the (t, u), (t', u') forms (besides which, q is unrestricted, except that it must be prime to p).

7. 2-*ic Criterion.*—It is known that *any one* of the five 2-ic forms of Art. 5*a*, 5*b* suffices to distinguish between $(q/p)_2 = \mp 1$) ;

[(1) *As to* (a, b).—This form is known to suffice for the 4-tic criterion, and therefore obviously suffices for the 2-ic criterion.

(2) *As to* (c, d), (e, f).—See Bickmore's paper "On the Numerical Factors of $(a^n - 1)$" in *Messenger of Math.*, Vol. XXVI, 1896–7, Arts. 18–20.

(3) *As to* (t, u), (t', u').—The mere non-existence or existence of these forms is itself the criterion.]

8. 4-*tic Criterion.*—It is known that *any one* of the three forms (a, b), (t, u), (t', u') suffices to distinguish between $(q/p)_4 = \mp 1$.

[See the author's paper "On 4-tic Residuacity, &c.," above quoted.]

8*a*. 4-*tic Criterion from* (a, b).—The law of 4-tic reciprocity, expressed by Gauss in terms of (a, b), is known to lead to a 4-tic criterion for all values of q. The deduction of the actual criterion for given values of q is unfortunately one of some difficulty, increasing rapidly as q increases.

8*b*. *Special Results.*—The results have been worked out for small values of q, and are published* in the following works :—

(1) *Mémoire sur les lois de réciprocité relatives aux résidus de puissances*, by P. Th. Pépin, S.J., Rome, 1878. This gives the results (on pp. 50–68) for the* small primes,

$$q = 2, \overline{3}, 5, \overline{7}, \overline{11}, 13, 17, \overline{19}, \overline{23}, 29 ;$$

(2) Two papers, "On the Numerical Factors of (a^n-1)," by the late Chas. E. Bickmore, in *Messenger of Math.*, Vol. xxv, 1895, pp. 15–38, and Vol. xxvi, pp. 2–9. This gives the results for

$$q = 2, \overline{3}, 5, \overline{6}, \overline{7}, 10, \overline{11}.$$

The formulæ given are all—(when q is an odd prime)—in the form

$$\mathfrak{a} \equiv \mu\mathfrak{b} \quad \text{or} \quad \mathfrak{b} \equiv \lambda\mathfrak{a} \pmod{q}. \tag{15}$$

8*c*. *4-tic Criterion from* (t, u), (t', u').—When $E = 4$, the general formula (14) reduces to

$$(\overline{q}/p)_4 = (t/p)_2 \div (u/p)_2, \qquad (q/p)_4 = (t'/p)_2 \div (u'/p)_2, \tag{16}$$

which may also be written

$$(\overline{q}/p)_4 = (tu/p)_2, \qquad (q/p)_4 = (t'u'/p)_2. \tag{16a}$$

Thus the evolution of a 4-tic criterion for any value of q whatever (prime, of course, to p) is reduced† to a simple reduction of 2-ic residues. The only difficulty in the use of this method is the production of the (t, u), (t', u') partitions themselves, which (when p or q is large) may be very great.

9. *8-vic Criterion from* (a, b), (c, d), (e, f).—A law of 8-vic reciprocity, expressed in terms of the (a, b), (c, d) partitions, has been evolved by‡ Herr Fr. Goldscheider. This should suffice, theoretically, for evolving an 8-vic criterion for any value of q whatever (prime, of course, to p). The actual deduction of the criterion in a rational form for given values of q is unfortunately one of considerable difficulty, increasing rapidly as q increases.

From the mutual interdependence of the (a, b), (c, d), (e, f) forms (Art. 5*b*), it should therefore be possible to evolve an 8-vic criterion for any base q whatever from any two of these partitions.

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* The author has extended this work to include all primes $q < 50$ and in such a form that the results for q = a product of such primes can be easily read off : it is hoped to publish this separately.

† For a fuller development of this, see the author's paper "On 4-tic Residuacity, &c.," above quoted.

‡ In *Das Reziprozitätsgesetz der achten Potenzreste*, in Programm No. 96 of the Luisenstadt Gymnasium, Berlin, 1889.

9*a*. *Special Results.*—The results in terms of the "2-ic parts" a, b, c, d were worked out by the late Mr. C. E. Bickmore for the small bases

$$q = 2, \bar{3}, 5, \bar{6}, 7, \overline{11}, 12,$$

and published in the two papers above quoted (Vol. xxv, p. 18; Vol. xxvi, pp. 15–22). Some of them seem to have been obtained in a somewhat tentative fashion, but have since been proved correct (except for two* misprints) by the law of 8-vic reciprocity above referred to, except that for $q = \overline{11}$ which required correction. The results for the small bases $q = 3, 5, 7, 11$ —(all taken +)—are quoted† in the table below (for purpose of comparison with the new criteria of the present paper).

In this use the even 2-ic parts (b, d) are reckoned +, and the *odd* 2-ic parts (a, c) are reckoned + or − as follows:

$$\text{a, c are + if } a = 4\alpha+1,\ c = 4\gamma+1\ ;\quad \text{a, c are } - \text{ if } a = 4\alpha-1,\ c = 4\gamma-1 \qquad (17)$$

q	$p-1$	Preliminaries.	Conditions for $(q/p)_8 = -1$	Conditions for $(q/p)_8 = +1$	Mod.
3	8ω	None.	$ac \equiv +1$	$ac \equiv -1$	3
	16ϖ	None.	$ac \equiv -1$	$ac \equiv +1$	3
5	8ϖ	$d = 10\delta$	$ac^3 \equiv -1$	$ac^3 \equiv +1$	5
		$c \pm 2d = 5i$	$2ac^3 \equiv -1$	$2ac^3 \equiv +1$	5
7	8ω	$b = 7\beta$	$cd \equiv 0$	$2c \pm d \equiv 0$	7
		$a = 7\alpha$	$2c \pm d \equiv 0$	$cd \equiv 0$	7
	16ϖ	$b = 7\beta$	$2c \pm d \equiv 0$	$cd \equiv 0$	7
		$a = 7\alpha$	$cd \equiv 0$	$2c \pm d \equiv 0$	7
11	8ω	$d(c \mp d) = 11i$	$(ac/11)_2 = +1$	$(ac/11)_2 = -1$	11
		$2c \mp d = 11i$	$(ac/11)_2 = -1$	$(ac/11)_2 = +1$	11
	16ϖ	$d(c \mp d) = 11i$	$(ac/11)_2 \equiv -1$	$(ac/11)_2 \equiv +1$	11
		$2c \mp d = 11i$	$(ac/11)_2 = +1$	$(ac/11)_2 \equiv -1$	11

10. 8-*vic Criterion Halved.*—Taking $E = 8$, whereby

$$p = 8\varpi+1,\quad (\bar{1}/p)_2 = (\bar{1}/p)_4 = +1,\quad (2/p)_2 = +1, \qquad (18)$$

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* On p. 17, l. 5, of Vol. xxvi, *read* $(\bar{6}/p)_8 = (\bar{1})^{\frac{1}{2}(a-1)+\frac{1}{8}(b+4)} \times bc \pmod 3$.

On p. 19, l. 8, of Vol. xxvi, *read* $(10/p)_8 = (\bar{1})^{\frac{1}{2}(a-1)+\frac{1}{8}(b-4)} \times bc(c^2-d^2) \pmod 5$.

† As revised and amended by Mr. Bickmore, with some further modifications by the present author, and some simplification (for case of $q = 11$) due to Dr. A. E. Western.

the preliminary conditions, $(\bar{q}/p)_2 = (q/p)_2 = +1$, lead easily to

$$(u/p)_2 = (u'/p)_2 = +1 \quad [\text{when } \lambda, \lambda' \text{ are squares}], \tag{18a}$$

so that the further preliminary 4-tic condition (16) gives

$$(\bar{q}/p)_4 = (t/p)_2 = +1, \quad (q/p)_4 = (t'/p)_2 = +1. \tag{18b}$$

The general formula (14) reduces to

$$(\bar{q}/p)_8 = (t/p)_4 \div (u/p)_4, \quad (q/p)_8 = (t'/p)_4 \div (u'/p)_4, \tag{19}$$

which, by aid of (18*a*), may be transformed to

$$(\bar{q}/p)_8 = (tu/p)_4, \quad (q/p)_8 = (t'u'/p)_4. \tag{19a}$$

The formal analogy with the 4-tic criteria (16), (16*a*) is worth notice. Hereby the evolution of an 8-vic criterion for any value of q whatever (prime to p) has been reduced to 4-tic work; and note that by (18*a*, *b*),

$$(t/p)_4 = \pm 1, \quad (u/p)_4 = \pm 1, \quad (t'/p)_4 = \pm 1, \quad (u'/p)_4 = \pm 1, \tag{20}$$

so that the imaginaries $\pm \iota$ are *not introduced*.

It remains to show how to reduce the 4-tic quantities involved in (19), (19*a*): two methods will be shown in Art. 10*a*, *b*.

10*a*. 8-*vic Criterion from* 2-*ic Work*.—The reduction of the 4-tic quantities in (19), (19*a*) may be done in the same way as that of $(q/p)_4$ in Art. 8*c*. For the preliminary conditions (Art. 2) are known [by (18*a*, *b*)] to be satisfied, so that it must be possible to express p in the forms

$$p = \frac{1}{\lambda_1}(t_1^2 \pm tu.u_1^2) = \frac{1}{\lambda_1'}(t_1'^2 \pm t'u'.u_1'^2), \tag{21}$$

with λ_1, λ_1' restricted like λ, λ' in Art. 5*a*, so that, by the process of Art. 8*c*,

$$(tu/p)_4 = (t_1u_1/p)_2, \quad (t'u'/p)_4 = (t_1'u_1'/p)_2, \tag{22}$$

and (19*a*) is hereby reduced to

$$(\bar{q}/p)_8 = (t_1u_1/p)_2, \quad (q/p)_8 = (t_1'u_1'/p)_2, \tag{23}$$

so that the 8-vic criterion is hereby made to depend *wholly on* 2-*ic work*. The process itself is thus of the simplest kind; but it has the practical disadvantage of requiring the production of one of the additional 2-ic partitions (21) with a large determinant ($\mp tu$ or $\mp t'u'$), which may be very laborious. The quantities $(t/p)_4$, $(u/p)_4$, ... in equation (19) might have been chosen for reduction instead of the $(tu/p)_4$, ... of equation (19*a*). Two auxiliary partitions would be required [instead of the single one used in (21)], but the determinants $(t, u, \ldots)$ involved would be much smaller.

10*b*. 8-*vic Criterion from* (a, b), (t, u), (t', u').—Reverting to the 8-vic criterion in the forms (19), (19*a*), it is known (Art. 8*a*) that the four 4-tic quantities therein involved can be computed from the single (a, b) partition. Hence the (a, b) partition with one of the (t, u), (t', u') partitions suffices for the evolving of an 8-vic criterion. And, from the mutual interdependence of the (a, b), (t, u), (t', u') forms (Art. 5*b*), it should also be possible to evolve an 8-vic criterion from any two of them together.

Although thus seen to be *possible*, it does not seem easy to evolve the criterion in a general way from these forms (even for particular* values of q).

11. *New* 8-*vic Criteria.*—In the absence of a general expression for the 8-vic criterion, in terms of the "2-ic parts" of any two of the five 2-ic partitions of Art. 5, it seemed worth while trying to obtain criteria in terms of the "2-ic parts" of these 2-ic partitions taken in pairs as follows:—

(a, b) and (c, d); (a, b) and (e, f); (a, b) and (t, u); (a, b) and (t', u'),

for each of the small prime bases

$$q = 3,\ 5,\ 7,\ 11,$$

from a comparison of the numerical values of the 2-ic parts for a large number of primes for which the actual (∓ 1) values of $(q/p)_8$ are known.

11*a*. *Criterion-Index* (μ).—It has been found convenient to express the criterion in the form

$$(q/p)_8 = (\bar{1})^{\mu}, \tag{24}$$

whereby $(q/p)_8 = -1$, when $\mu = \omega$, and $= +1$, when $\mu = \epsilon$, (24*a*)

so that the criterion depends only on the *oddness* or *evenness* of μ, and it suffices to seek its expression in terms of the "2-ic parts." This quantity may be called the *criterion-index*.

It was found that—(for these four small bases)—the index (μ) takes very simple forms, depending only on the linear forms of p to modulus 8, and of the 2-ic parts to modulus 2, 4, q, $2q$, $4q$, &c.

Writing p and the "2-ic parts" in the following forms, wherein r, r'

* A general expression could only be expected in terms of the imaginary 4th roots of $\bar{1}$.

are certain *residues* to mod 4, $2q$, $4q$,

$$p = 8m+1, \tag{25a}$$

$$\mathrm{a} = 2q\alpha + r_\alpha, \quad \mathrm{c} = 2q\gamma + r_\gamma, \quad \mathrm{e} = 2q\eta + r_\eta, \tag{25b}$$

$$\mathrm{b} = 4\beta, \qquad \mathrm{d} = 2\delta. \qquad \mathrm{f} = 2\phi, \tag{25c}$$

$$t = 4\tau \pm 1, \text{ or } \lambda q\tau + r_\tau, \quad \text{where} \quad \lambda = 2, 4, \tag{25d}$$

$$u = 4\upsilon + r_\upsilon, \text{ or } q\upsilon + r_\upsilon, \tag{25e}$$

$$t' = 4\tau' + r'_\tau, \text{ or } \lambda' q\tau' + r'_\tau, \quad \text{where} \quad \lambda' = 2, 4, \tag{25f}$$

$$u' = 4\upsilon' \pm r'_\upsilon, \text{ or } qu' + r'_\upsilon, \tag{25g}$$

then the index (μ) is expressible in the four simple formulæ

μ = sum *of some* of, or *all of*, the five quantities in brackets,

$$[\alpha+\beta+\gamma+\delta+m], \quad [\alpha+\beta+\eta+\phi+m], \tag{26a}$$

$$[\alpha+\beta+\tau+\upsilon+m], \quad [\alpha+\beta+\tau'+\upsilon'+m]. \tag{26b}$$

11*b*. 8-*vic Criterion Table.*—The actual formulæ obtained for the four bases $q = 3, 5, 7, 11$ are shown in the tables below. Table I shows the linear forms in which the ten 2-ic parts a, b, c, d, ... must be expressed so as to yield the auxiliaries α, β, γ, δ, ... required in the μ-formulæ. Table II gives the final reduced formulæ for the criterion-index (μ).

These tables were drawn up from an examination of the known (∓ 1) values of $(q/p)_8$ for $q = 3, 5, 7, 11$ for all primes $\not> 10000$. The number of test-primes available—shown in abstract below—

Base q =	3	5	7	11
Number of Test-primes for $(q/p)_8 = -1$	38	36	35	31
Number of Test-primes for $(q/p)_8 = +1$	28	33	37	40
Total ...	66	69	72	71

was so large that there could be little doubt of the correctness of the formulæ.

The results were, of course, *tentative;* but the theory of the subject has—since this paper was written—been investigated* by Dr. A. E. Western. The two Tables (I, II) have been thoroughly revised† by him, so that the formulæ may now be accepted as *proved.*

* This will probably be presented by him in a separate paper.

† Besides several simplifications in the tables, he has kindly supplied the formulæ for $(11/p)_8$ in terms of (a, b), (e, f)—(in which the author had failed)—and also some corrections in those for $(3/p)_8$ and $(7/p)_8$ in terms of (a, b), (t', u')].

Table I.

Linear Forms of p, a, b, c, d, e, f, t, u, t', u' *for* 8*-vic Criterion.*

q	p	$p-1$	a	b	c	d	e	f	t	u	t'	u'
3	$6\varpi+1$	$8m$	$6\alpha\pm1$,	4β	$6\gamma\pm1$,	2δ	$6\eta+1, 3, 5,$	2ϕ	$12\tau\pm1$,	4υ	$2\tau'+1$,	$4\upsilon'$
											$4\tau'+2$,	$4\upsilon'\pm1$
5	$10\varpi+1$	$8m$	$10\alpha\pm1$,	4β	$10\gamma\pm1, 3,$	2δ	$10\eta\pm1, 3,$	2ϕ	$10\tau\pm1$,	4υ	$10\tau'\pm1$,	$4\upsilon'$
	$10\varpi-1$	$8m$	$10\alpha\pm3$,	4β	$10\gamma\pm3, 9,$	2δ	$10\eta\pm1, 7,$	2ϕ	$20\tau\pm2$,	$u=\upsilon=\omega$	$10\tau'\pm3$,	$4\upsilon'$
7	$7\varpi+1$	$8m$	$7\alpha\pm0, 1,$	4β	$7\gamma\pm0, 1, 2,$	2δ	$14\eta\pm1, 11,$	2ϕ	$28\tau\pm1$,	4υ	$4\tau'\pm1$,	$4\upsilon'$
	$7\varpi+2$	$8m$	$7\alpha\pm0, 3,$	4β	$7\gamma\pm0, 3, 6,$	2δ	$14\eta\pm3, 5,$	2ϕ	$28\tau\pm3$,	4υ	$4\tau'$,	$u'=\omega$
	$7\varpi+4$	$8m$	$7\alpha\pm0, 5,$	4β	$7\gamma\pm0, 4, 5,$	2δ	$14\eta\pm9, 13,$	2ϕ	$28\tau\pm9$,	4υ		
11	$22\varpi+R$	$8m$	$22\alpha\pm1, 5, 7, 9, 19,$	4β	$22\gamma\pm1, 5, 7, 9, 19,$	2δ	$e=\omega,$	2ϕ	$t=2\omega$,	8υ	$2\tau'+1$,	$4\upsilon'$
									$4\tau\pm1$,	$4\upsilon\pm1$	$4\tau'+2$,	$4\upsilon'\pm1$

Table II.

Value of 8*-vic Criterion-Index* (μ) *in* $(q/p)_8=(\overline{1})^{\mu}$.

q	By (a, b) and (c, d) μ	By (a, b) and (e, f) μ	By (a, b) and (t, u) μ	By (a, b) and (t', u') μ
3	$\alpha+\gamma+m$	$\alpha+\eta+m$	$\alpha+\upsilon$	$[t'=\omega]$, $\alpha+\beta+\tau'+\upsilon'+m$ $[t'=\epsilon]$, $\alpha+\beta+\tau'+m$
5	$\alpha+\gamma$	$\alpha+\eta$	$\alpha+\beta+\tau+\upsilon$	$\alpha+\tau'+\upsilon'+m$
7	$[c=7\gamma]$, $\alpha+m$ $[c\neq7\gamma]$, $\alpha+\gamma+m$	$\alpha+\beta+\eta+\phi$	$\alpha+\upsilon$	$[u'=\epsilon]$, $\alpha+\beta+\upsilon'+m$ $[t'=\epsilon]$, $\alpha+\beta+\tau'+m$
11	$[d(c^2-d^2)=11i]$, $\alpha+\gamma+m$ $[(2c\pm d)=11i]$, $\alpha+\gamma+m+1$	$[f(f^2-4e^2)=11i]$, $\alpha+\beta+\phi$ $[e(f^2-9e^2)=11i]$, $\alpha+\beta+\phi+1$	$[tu=\epsilon]$, $\alpha+\upsilon$ $[tu=\omega]$, $\alpha+\tau+m$	$[u'=\epsilon]$, $\alpha+\beta+\tau'+\upsilon'+m$ $[u'=\omega]$, $\alpha+\beta+\tau'+m$

12. *Special Primes* (8-vic Criterion).—Primes of the following forms yield readily sufficient data for forming the 8-vic criteria, inasmuch as they yield the requisite 2-ic partitions *algebraically*.

i $p = x^4 + y^4$;
ii $p' = \frac{1}{2}(x^4 + y^4)$;
iii $p = (x^{2q} + y^{2q}) \div (x^2 + y^2)$;
iv $p' = \frac{1}{q}(x^{2q} + y^{2q}) \div (x^2 + y^2)$
v $p = \frac{1}{\lambda}(x^4 \pm qy^4)$.

It is supposed, of course, that the preliminary 4-tic conditions (4), (5) are satisfied.

12*a*. *Forms* i, ii. $p = (x^4 + y^4), \quad p' = \frac{1}{2}(x^4 + y^4).$ (27)

These can be expressed (algebraically) in the three 2-ic partitions (a, b), (c, d), (e, f), from which the 8-vic criterion is known to be derivable (Art. 9). The resulting expression in x, y may be expected to be much simpler than the general expression of Art. 9. In one case it takes quite simple forms, thus

$$y = 2mq \text{ gives } (q/p)_8 = +1, \tag{28a}$$

$$x \not\equiv y = 2mq \text{ or } 4mq \text{ gives } (\pm q/p')_8 = +1, \text{ when } q = 4k \pm 1, \tag{28b}$$

are sufficient, but by no means always necessary, conditions.

12*a'*. *Special Results.*—The criteria for these two forms (27) for the small bases

$$q = \pm 2, 3, 5, 6, 7, 10, 11, 12,$$

have been worked out by the author, and published in his paper "On High Quartan Factorisations," &c., in *Messenger of Mathematics*, Vol. 36, 1907, Art. 18, and Table A. The results are all of very simple form.

A list of *all* primes of these two forms ($\not> 9 \cdot 10^6$), and also of *all* high primes of same forms, in which $x = 1$ (up to $p \not> 2 \cdot 10^9$, $p' \not> 2\frac{1}{4} \cdot 10^9$) will be found in Tables I–IV of the same paper.

12*b*. *Forms* iii, iv. [q *odd.*]

$$p = (x^{2q} + y^{2q}) \div (x^2 + y^2), \qquad p' = \frac{1}{q}(x^{2q} + y^{2q}) \div (x^2 + y^2). \tag{29}$$

These forms can be expressed (algebraically) in the three 2-ic partitions (a, b), (t, u), (t', u'), from which the 8-vic criterion is known to be derivable (Art. 10*b*).

12*b'*. *Special Results.*—General expressions for the 8-vic criteria cannot be conveniently written down for these two forms (29); but, for any

*given** primes of these forms, the 8-vic criteria for the four small bases $q = 3, 5, 7, 11$ can be quite easily formed from the formulæ and table of Art. 11*b*.

12*c*. *Forms* v. $$p = \frac{1}{\lambda}(x^4 \pm qy^4), \tag{30}$$

These forms give at once, by the same process as in Art. 8*c*,

$$(\mp q/p)_8 = (xy/p)_2, \tag{31}$$

whereby the finding of the 8-vic criterion has been reduced to 2-ic residue-work.

13. 16-*ic Criterion halved.*—Taking $E = 16$, whereby

$$p = 16\varpi + 1, \quad (\bar{1}/p)_2 = (\bar{1}/p)_4 = (\bar{1}/p)_8 = +1, \quad (2/p)_2 = +1, \tag{32}$$

the conditions $(\bar{q}/p)_2 = (q/p)_2 = +1$, $(\bar{q}/p)_4 = (q/p)_4 = +1$, give, as in Art. 10,

$$(t/p)_2 = (u/p)_2 = (t'/p)_2 = (u'/p)_2 = +1, \tag{32a}$$

so that the further 8-vic condition (19) gives

$$(t/p)_4 = (u/p)_4, \qquad (t'/p)_4 = (u'/p)_4, \tag{32b}$$

and the general formula (14) becomes

$$(\bar{q}/p)_{16} = (t/p)_8 \div (u/p_8), \qquad (q/p)_{16} = (t'/p)_8 \div (u'/p)_8, \tag{33}$$

which is formally analogous to (19), but cannot generally be reduced to forms similar to (19*a*) [because $(t/p)_4$, $(u/p)_4$, ..., are not necessarily $= +1$].

The finding of a 16-ic criterion has thus been reduced to 8-vic residuacity work. The 8-vic quantities involved in equation (25) may now be reduced by any of the processes explained in Art. 9, 10, 10*a*, *b*, the relative advantages of which may be thus contrasted :

(1) By the method of Art. 10*a*, the work will depend wholly on 2-ic residue work, requiring however several fresh 2-ic partitions (of large determinant).

(2) By the method of Art. 10*b*, only two 2-ic partitions of p (both of large determinant), in addition to the (a, b) partition, are still required. Although the data will then be *sufficient*, the practical difficulty of the final reduction is still considerable.

(3) By the method of Art. 9, the reduction of both the 8-vic quantities in (33) should be possible by the aid of two out of the (a, b), (c, d), (e, f) partitions. The reduction by the law of 8-vic reciprocity will be one of considerable difficulty.

* Lists of primes of these forms for the indices $2q = 6, 10, 14, 18, 22, 26, 30$ will be found in a series of Tables, &c. by the author (now being published).

It would seem that—until some simplification of the modes of reduction of the 4-tic and 8-vic laws of reciprocity is made—the method of Art. 10*a* is not only the most elementary, but also the easiest of application. It seems probable, however, that in many cases the direct solution of the 2-ic congruences of Art. 4 would be the simplest mode available.

13*a*. *Simple Case.*—When both t, u, or both t', u' are such that their 8-vic criteria are *known* in terms of (a, b), (c, d), the reduction of (33) can be at once effected: this involves (at present) that

t, u, t', u' should be = some of $\pm$ 2, 3, 5, 6, 7, 10, 11, 12 or powers thereof.

14. *Special Primes* (16-ic Criterion).—Primes of the following forms yield sufficient data for the 16-ic criterion, viz.,

i $p = x^8 + y^8$ | ii $p' = \frac{1}{2}(x^8 + y^8)$ | iii $p = (x^{4q} + y^{4q}) \div (x^4 + y^4)$ | iv $p' = \frac{1}{q}(x^{4q} + y^{4q}) \div (x^4 + y^4)$ | v. $p = \frac{1}{\lambda}(x^4 \pm qy^4)$.

It is supposed, of course, that the preliminary 8-vic conditions (4), (5) are satisfied.

14*a*. *Forms* i, ii. $p = (x^8 + y^8)$, $p' = \frac{1}{2}(x^8 + y^8)$. (34)

The 8-vic criteria can, of course, be derived from these forms, being only special cases of Forms i, ii of Art. 12*a*. The 16-ic criteria derivable from them have not been fully worked out: but, from such data as are available, it seems certain that results similar* to (28*a*, *b*) obtain with them also, viz.,†

$$y = 2mq \quad \text{gives} \quad (q/p)_{16} = +1, \tag{34a}$$

$$x \pm y = 4mq \text{ gives } (\pm q/p')_{16} = +1, \text{ when } q = 4k \pm 1, \tag{34b}$$

and, it seems further probable that,

$$\text{if} \quad (q/p_4)_8 = \phi(x, y), \quad \text{and} \quad (q/p'_4)_8 = \psi(x, y), \tag{34c}$$

be the 8-vic criteria for primes of the forms (27), then the 16-ic criteria of q for primes of the forms (34) are* of the *same form*, viz.,

$$(q/p)_{16} \quad \phi(x, y), \qquad (q/p')_{16} = \psi(x, y). \tag{34d}$$

* See Art. 20 of the author's paper on "High Quartan Factorisations," &c. above quoted. There are remarkably few primes of these two forms [the reason being that $(x^8 + y^8)$ is usually divisible by 17]: only four of each kind ($< 10^9$) being known (to the author), see Table VII of the paper quoted for a list of these.

† The two Results (34*a*), (34*b*) have since been proved by Dr. A. E. Western to follow directly from his form of the Law of 16-ic Reciprocity.

14*b*. *Forms* iii, iv ; [*q odd*].

$$p = (x^{4q} + y^4) \div (x^4 + y^4), \qquad p = \frac{1}{q}(x^{4q} + y^{4q}) \div (x^4 + y^4). \tag{35}$$

These primes* can be expressed (algebraically) in each of the five partitions (a, b), (c, d), (e, f), (t, u), (t', u'), from which the 16-ic criterion is known to be derivable (Art. 13).

14*c*. *Form* v. $$p = \frac{1}{\lambda}(x^8 \pm qy^8). \tag{36}$$

These forms give at once, by the same process as in Art. 8*c*,

$$(\mp q/p)_{16} = (xy/p)_2, \tag{37}$$

whereby the finding of the 16-ic criterion has been reduced to 2-ic residue-work.

* Very few primes of these two forms are known.

ON A NEW METHOD IN THE THEORY OF INTEGRATION

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1. *Introduction.* — Though the importance of the concept of a generalised integral as originally discovered by Lebesgue is now fully recognised, the ideas involved are still so unfamiliar to the majority of mathematicians that almost all the work on the subject has consisted of deductions from that concept of the consequences that naturally follow. No attempt, as far as I know, other than my own, commenced originally at a period when I was unacquainted with Lebesgue's work, have been made towards a presentation, in a different order, of the ideas involved.

Among the apparently minor differences between my own work and that of Lebesgue, that which concerns the definition of the content of an open set has not escaped notice; it has even been argued that my definition is less natural than that of Lebesgue. The difference in question is that I define the inner content of an open set of points as the upper bound of the content of closed sets contained in that open set. To myself the naturalness of this definition scarcely requires vindication, in view of the fact that it is precisely with the closed sets contained in the open set that all work anterior to that of a few years ago concerns itself. For example, the points of discontinuity of a function form a set which is itself the outer limit of a closed set, and in this connection I may, perhaps, venture to refer to a small note of mine in the *Quarterly Journal of Mathematics** as bearing on this point. For the rest, in the case of a set whose region of existence is infinite, the definition in question is the most natural, as well as the most convenient, the more so that we have now convinced ourselves that all sets of points which can present themselves mathematically are bound to have the same inner and outer content.

* "A Note on the Condition of Integrability of a Function of a Real Variable," *Quarterly Journal of Mathematics*, 1903, Vol. xxxv, pp. 189–192. See also Hobson's *Theory of Functions of a Real Variable*, p. 342.

This definition, taken in conjunction with the known theorem,* that the ordinate section of a closed plane set of points represents an upper semi-continuous function of the abscissa, itself suggests to the attentive reader a different method of treatment of the theory of integration. This method, like another indicated by myself, in which the interval of integration is divided into measurable sets, and the Darboux-Riemann method applied to these sets and the upper and lower bounds of the function in them, has for its object the emphasizing of the naturalness of the generalisation. The method I am about to give has, however, the further advantage that it enables one to grasp intuitively the *a priori* inevitableness of the fact, that many theorems true for ordinary integrals must be true for the generalised integrals. That this was bound to be the case was indeed tacitly assumed by some writers, and it has only recently been found necessary to give detailed proofs. These proofs employ methods already used with success by Lebesgue in his earlier work. Though, however, they are easily followed by an expert, they are far from being at the first glance of an intuitive character.

In the present paper all turns on the use of monotone sequences. The systematic use of such sequences constitutes a powerful weapon in analysis. Coupled with the theorem of Weierstrass that any continuous function can be expressed as the uniform limit of a polynomial, and in particular therefore as the uniform limit of a differentiable function, we can reduce many of the theorems of analysis that we may require to considerations of the simplest possible character. Since the inner content of a set is the upper bound of the content of closed sets contained in the set, it follows that the generalised lower integral is the upper bound of the integrals of upper-semi-continuous functions not greater than the function, in the case in which the function is positive (*i.e.*, $\geqslant 0$) and bounded. It at once follows, by subtracting the function from a constant function, that the upper generalised integral is the lower bound of the integrals of lower semi-continuous functions not less than the function. We are thus led in the case of a bounded summable function to see that, for purposes of integration, such a function may be replaced indifferently by

(1) the limit of a monotone ascending sequence of upper-semi-continuous functions, or

(2) the limit of a monotone descending sequence of lower-semi-continuous functions,

* W. H. Young, "The General Theory of Integration," 1904, *Phil. Trans.*, Series A, ccIv, 1905, pp. 221–252. Young's *Theory of Sets of Points*, 1906, p. 256, § 163.

that is, by functions which I have called respectively lower-upper and upper-lower semi-continuous functions, or shortly *lu*-functions and *ul*-functions.

I have already in a previous paper defined the integrals of upper and lower semi-continuous functions by the method of monotone sequences, and it is not difficult to reconstruct the whole theory of generalised integration, *de novo*, taking as the point of departure the definition of the integral of a continuous function, or still more simply, if we employ the Weierstrassian sequence, the definition of the integral of a positive integral power of x, or of the product of positive integral powers of x and y, &c., and then defining successively, by means of monotone sequences, the integrals of semi-continuous functions, and of *lu*-functions and *ul*-functions, and therefore of any bounded function.

The case of unbounded functions requires certain modifications, as in other modes of treatment. We may, however, for the purposes of integration replace an unbounded function either by what I have called a *lul*-function or an *ulu*-function, that is, by the limit of a monotone ascending or descending sequence of *ul*-functions or *lu*-functions respectively.

It is, in fact, one of the advantages of our method that the step from bounded to unbounded functions is of the same nature as that taken when passing from continuous functions to bounded discontinuous functions, so that the processes employed are homogeneous throughout.

In the present paper I propose rather to shew how the use of the new definition, or of the properties underlying it, simplifies the proofs of the fundamental theorems of the integral calculus, than to attempt a reconstruction of the theory. For this purpose I have selected the connection between repeated integration and double integration, the method of integration by substitution, the Second Theorem of the Mean, and an inequality, known in its original form as Schwarz's inequality, of which important use has been recently made by Lebesgue.

I take occasion to obtain, at the same time, certain properties of parametric integrals of unbounded functions, which I believe to be new and not without importance. It will be noticed that they are in part extensions of previous results obtained by myself. Among such I may perhaps specially mention the theorem that, if v is an essentially positive not necessarily bounded function of (x, y), whose integral with respect to x is a continuous function of y, and u is any bounded measurable function of x whatever, the integral of uv with respect to x necessarily belongs to Baire's first class, and is therefore pointwise discontinuous with respect to every perfect set of values of y. This is, indeed, an immediate conse-

quence of the theory and of the fact, here pointed out, that *it is a necessary and sufficient condition for a function to belong to Baire's first class that it should be both a lu-function and also an ul-function.*

Though I have assumed certain results of previous papers, reference to which is made, I have tried, as far as possible, to render the account here given in other respects complete in itself, and intelligible to persons unacquainted with those papers.

In conclusion, it may not be out of place to remark that the success of the method depends on two apparently antagonistic facts :—

(1) that certain properties are preserved in the passage from a function of a monotone sequence to the limiting function of that sequence ;

(2) that certain other properties may be partially or entirely lost.

It is in virtue of the former circumstance that term-by-term integration is allowable, provided the integrals of the constituent functions are finite. The latter fact enables us to express functions which are themselves destitute of certain properties, *e.g.*, differentiability, expansibility in Taylor's series, continuity, as the limits of monotone sequences of functions possessing the property in question. This is remarkable ; it is, however, in accordance with the sum total of human experience, which has taught us to draw advantage from what, at the outset, is found to be a hindrance, but conditions none the less the possibility of progress.

2. The functions which constantly occur in the following investigation are semi-continuous functions, and the limits of monotone sequences of semi-continuous functions, and of the functions so formed. The reader is reminded that the limit of a monotone ascending (descending) sequence of lower (upper) semi-continuous functions is a lower (upper) semi-continuous function.* Thus the less familiar functions now to be considered are :

(1) functions which are the limits of ascending sequences of upper semi-continuous functions ; these will be called *lower-upper semi-continuous functions*, or shortly, *lu-functions :*

(2) functions which are the limits of descending sequences of lower semi-continuous functions ; these will be called *upper-lower semi-continuous functions*, or shortly *ul-functions.*

In using this notation we are guided by analogy with what takes place in the case of monotone sequences of continuous functions. If the

* W. H. Young, "On Functions defined by Monotone Sequences and their Upper and Lower Bounds," 1908, *Messenger of Math.*, Vol. XXXVII, p. 152.

sequence is *ascending* the limit is characterised by the prefix *lower*, and if it is *descending* by the prefix *upper*. Using this mode of characterising functions we have the following statement of the facts above quoted concerning monotone sequences of semi-continuous functions :—

an upper-upper semi-continuous function is an upper semi-continuous function ;

a lower-lower semi-continuous function is a lower semi-continuous function.

Since $f(x)+e$, where $e<0$ or $e>0$, is upper (lower) semi-continuous, provided $f(x)$ is so, it is obvious that we have the following :—

a lower (upper) semi-continuous function is a special case both of a lu-function and also of an ul-function.

3. It will be remembered, or is easily proved, that semi-continuous functions have the following three properties, which it will be found are retained by the functions here introduced :—

(1 & 2) *the sum and the product of two upper (lower) semi-continuous functions is a function of the same type, provided, in the case of the product, both functions are positive, that is* $\geqslant 0$;

(3) *the function whose value at each point is the greater of the values of two upper (lower) semi-continuous functions, or the less of the values of two upper (lower) semi-continuous functions, is a function of the same type.*

The proofs of these theorems usually given depend on the defining properties of semi-continuous functions. The reader may, however, be reminded that it has been proved* that *any upper (lower) semi-continuous function with a finite upper (lower) bound is the limit of a monotone sequence of continuous functions*, so that this property might have been taken as the definition of these functions, in which case the proofs of the properties cited would have been identical in form with that to be now given for our new functions, a form of proof which, being recurrent, is immediately applicable to the functions of the successive classes subsequently to be formed on the same principle.

It may here be pointed out, *en passant*, as a fact which will be useful in the sequel (§ 32), that the constituent functions of the sequence, representing, as they do in the given construction, polygons, have right-hand

* W. H. Young, "On Monotone Sequences of Continuous Functions," 1908, *Proc. Camb. Phil. Soc.*, Vol. XIV, p. 523.

and left-hand differential coefficients at every point, and these, being constant in each of a finite number of intervals into which the fundamental segment has been divided, are continuous except at a finite number of points.

Returning to the main subject of the present article, we remark that the proof of all three properties is of the same form. Let $a(x)$ and $A(x)$ denote two *lu*-functions, and let $a_n(x)$ and $A_n(x)$ denote the constituent upper semi-continuous functions of the monotone ascending sequences defining $a(x)$ and $A(x)$ respectively.

Let $s_n(x)$ denote the upper semi-continuous function formed from $a_n(x)$ and $A_n(x)$ by combination in accordance with one of the properties 1, 2, and 3 above cited, and $s(x)$ the function formed by the same rule from $a(x)$ and $A(x)$. Then it is clear that the functions $s_n(x)$ form a monotone ascending sequence, and that their limit is $s(x)$, which proves the theorem in this case. Similarly it may be proved in the other cases.

4. It is on the property (3) of semi-continuous functions, signalised in the preceding section, that the proof of the following theorem depends.

THEOREM.—*The limit of a monotone ascending (descending) sequence of lu-functions (ul-functions) is a function of the same type.*

For let

$a_{11} \leqslant a_{12} \leqslant a_{13} \leqslant \ldots$ be upper semi-continuous functions having as limit a_1,

$a_{21} \leqslant a_{22} \leqslant a_{23} \leqslant \ldots$,, ,, ,, ,, a_2,

and so on. Then, if $a_1 \leqslant a_2$,

we can ensure that for every index n,

$$a_{1n} \leqslant a_{2n}.$$

For, if this is not the case we have only to replace a_{2n} by the function whose value at any point is the greater of the values of a_{1n} and a_{2n}. The new function is still upper semi-continuous, and at each point is not greater than either a_1 or a_2, and therefore certainly not greater than a_2. Also the monotony of the sequence is clearly unaffected, so that this device serves the purpose required.

This being so, it is evident that if a is the limit of an ascending sequence of *lu*-functions, and we express each of these as the limit of an ascending sequence of upper semi-continuous functions, we have a system of functions forming a table as follows, the functions in each row and

column forming a monotone ascending sequence :—

$$\begin{array}{ccccccc}
a_{11} & \leqslant & a_{12} & \leqslant & a_{13} & \leqslant \ldots \leqslant & a_1, \\
\wedge & & \wedge & & \wedge & & \wedge \\
a_{21} & \leqslant & a_{22} & \leqslant & a_{23} & \leqslant \ldots \leqslant & a_2, \\
\wedge & & \wedge & & \wedge & & \wedge \\
a_{31} & \leqslant & a_{32} & \leqslant & a_{33} & \leqslant \ldots \leqslant & a_3, \\
\ldots & & \ldots & & \ldots & & \wedge \\
\ldots & & \ldots & & \ldots & & a.
\end{array}$$

From which it is evident that a is the limit of the monotone ascending sequence of upper semi-continuous functions

$$a_{11} \leqslant a_{22} \leqslant a_{33} \leqslant \ldots,$$

which proves the theorem in the case of an ascending sequence of *lu*-functions. Similarly the other case may be proved.

5. If therefore we denote the limit of a monotone sequence of lower-upper (upper-lower) semi-continuous functions by the same term preceded by lower or upper according as the sequence is ascending or descending, so that the principle of the notation agrees with that already used, and with the facts known about limits of monotone sequences of continuous functions, we have the following mode of stating the preceding theorem :—

a lower-lower-upper semi-continuous function is a lower-upper semi-continuous function :

an upper-upper-lower semi-continuous function is an upper-lower semi-continuous function.

We also have two new types of function, viz., *a lower-upper-lower and an upper-lower-upper semi-continuous function*, or, shortly, *a lul-function and an ulu-function.*

6. It is now clear that this system of functions may be continued. We shall have the same three theorems about *lu*-functions and *ul*-functions as those quoted in § 3 for semi-continuous functions, viz., the *sum and product of two lu-functions or of two ul-functions is a function of the same type* (*provided, in the case of the product, both functions are positive*), *and so is the function formed by taking at each point the greater, or the less, of the values of the two lu-functions* (*ul-functions*).

Hence, as before,

an upper-upper-lower-upper (lower-lower-upper-lower) semi-continuous function is an upper-lower-upper (lower-upper-lower) semi-continuous function.

Thus we again get two new types of function, viz., *a lower-upper-lower-upper and an upper-lower-upper-lower semi-continuous function.*

This is perfectly general; we get two new functions at each stage characterised by the alternate use of the words upper and lower, while the word upper (lower) used twice consecutively is equivalent to its use once only.

7. Still more compactly we may call the successive pairs of functions formed in this manner

an l and a u,
a lu and an ul,
a lul and an ulu,
a lulu and an ulul, &c.,

dropping the word "function" altogether.

Thus we have the following relations, which are at once evident:—

an l and a u are each special cases both of a lu and also of an ul,

a lu and an ul are both special cases both of a lul and also of an ulu, and so on.

Again:

(1 & 2) *the sum and the product of two lu-functions (or of two ul-functions) is a function of the same type, provided, in the case of the product, both functions are positive;*

(3) *the function whose value at each point is the greater, or the less, of the values of two lu-functions (ul-functions) is a function of the same type.*

From these results we have, by the same method, similar results for the pair of types of function belonging to the next stage, viz.:—

(1 & 2) *the sum and product of two ulu-functions (or of two lul-functions) is a function of the same type, provided, in the case of the product, both functions are essentially positive;*

(3) *the function whose value at each point is the greater, or the less, of the values of two ulu-functions (lul-functions) is a function of the same type.*

It is clear that these theorems may be stated for the pair of types of function belonging to any stage. We shall not insist more on this here, since, as will be seen in the sequel (§ 22), for purposes of integration, we need not go beyond *lul*'s and *ulu*'s.

8. The following properties of our functions are almost immediately obvious :—

If we change the sign of a lu-function, it becomes an ul-function, and vice versa.

Generally the change of sign changes a *lul*-function into a *ulu*-function, and so on, the letters u and l being throughout interchanged.

THEOREM.—*If $f(x)$ is a continuous or an upper (lower) semi-continuous function, or if it is a lu (or an ul), or an ulu (or a lul), &c., and $f(x, M)$ is the function got from $f(x)$ by retaining at any point the value of $f(x)$ if less (greater) than M, and at other points changing the value to M, $f(x, M)$ is a function of the same type as $f(x)$.*

This follows immediately from property (3) of the preceding article, since a constant is a special case of any of our types of function.

9. In a paper already quoted I shewed that a monotone decreasing (increasing) sequence of functions which are upper (lower) semi-continuous at a point P has for limit a function which is also upper (lower) semi-continuous at P. Moreover, the argument held whether we were referring to the continuum or to any perfect set.

Now a semi-continuous function is continuous with respect to the continuum or to any perfect set at points which are dense everywhere in the continuum or in that perfect set, since it is a function of Baire's* first class. Hence, in considering a monotone increasing sequence of upper semi-continuous functions, there are points in every interval at which the

* R. Baire, "Sur les fonctions de variables réelles," 1899, *Ann. di Mat.* Ser. 3, Vol. III, pp. 1–123. Other proofs of Baire's theorem have been given by the author, "A New Proof of a Theorem of Baire's," 1907, *Messenger of Math.*, Vol. XXXVII, pp. 49–54. See also "A Note on Functions . . . which assume all values between their Upper and Lower Bounds," 1909, *Messenger of Math.*, Vol. XXXIX, pp. 69–72, *et al.*

functions are continuous, and therefore lower semi-continuous, so that the limiting function is lower semi-continuous there. Since this is true also with respect to any perfect set, we have the following theorem :—

*A lu- (ul-) function is lower (upper) semi-continuous with respect to the continuum or to any perfect set at points which are dense everywhere in the continuum or in that perfect set.**

In particular, we have the following corollary :—

A function which is both a lu and an ul is pointwise discontinuous with respect to the continuum or any perfect set, and is therefore a function of Baire's first class, that is, it is the limit of a continuous function.

In the next section it will be seen that the converse of this is also true.

10. In another paper† I discussed the properties of those functions which at every point have as value the upper, or the lower, bound of the values at that point of a succession of continuous, or of semi-continuous, functions. These functions I called *the upper function and the lower function of the succession.* Among these properties we have the following, which is here stated with the new nomenclature :—

(1) *The upper (lower) function of a succession of lower (upper) semi-continuous functions is an ul-function (lu-function).*

Hence, taking such a succession that the upper and lower functions coincide in an unique limiting function, in which case the succession is called *a sequence*, we have the following theorems :—

(1) *A function which is the unique limit of a sequence of lower (upper) semi-continuous functions is an ul-function (lu-function).*

(2) *A function of Baire's first class (that is, the unique limit of a sequence of continuous functions) is both an ul and also a lu.*

Combining this last theorem with the result at the end of the preceding section, we have the following result :—

The necessary and sufficient condition that a function should belong to Baire's first class is that it should be both an ul and also a lu.

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* The remaining points form, of course, a set of the first category.

† W. H. Young, "Oscillating Successions of Continuous Functions," 1908, *Proc. London Math. Soc.*, Ser. 2, Vol. 6, pp. 298–320.

11. A lower generalised integral of a bounded function $f(x)$ which is $\geqslant 0$ is, as we know, the inner content of a measurable set of blocks (ordinates).* Let the range of integration be the segment

$$0 \leqslant x \leqslant c,$$

and let this inner content be $I(c)$, or shortly I.

We also know that I is the upper bound of the content of closed sets contained in the set of blocks, of which I is the inner content.†

Again, the content of a closed plane set is the generalised integral of the content of its ordinate section; moreover, the content of its ordinate section is an upper semi-continuous function of x, and therefore its generalised integral is its ordinary Darboux upper integral.‡

Now the ordinate section of a closed plane set contained in the set of blocks is a closed set lying inside the corresponding block, and is therefore not greater than the length of the block (ordinate). This shews that $I(c)$ is the upper bound of the integrals of certain upper semi-continuous functions $\leqslant f(x)$, so that $I(c) \leqslant$ upper bound of integrals of all upper semi-continuous functions $\leqslant f(x)$. On the other hand, the integral of any upper semi-continuous function $\leqslant f(x)$ is the content of a closed set of blocks inside the original set of blocks and is therefore $\leqslant I(c)$. For, if $g(x) \leqslant f(x)$ be such a function, $g(x) \geqslant$ the upper limit of its values in the neighbourhood. Hence any sequence of points lying inside blocks in the neighbourhood and having their limiting point on the ordinate at the point x, being taken, that limiting point will lie inside the block at x. Hence the upper bound in question $\leqslant I(c)$. Thus we get the following theorem :—

The generalised lower integral of a positive bounded function $f(x)$ is the upper bound of the integrals of upper semi-continuous functions $\leqslant f(x)$.

12. Next let $f(x)$ be any bounded function, and let K be a positive number such that $K+f(x)$ is a positive function. Then, if $g(x)$ is any upper semi-continuous function $\leqslant f(x)$, $K+g(x)$ is an upper semi-continuous function $\leqslant K+f(x)$, and conversely, if $K+g(x)$ is an upper semi-continuous function $\leqslant K+f(x)$, $g(x)$ is an upper semi-

* *The General Theory of Integration*, p. 247, § 28.
† Young's *Theory of Sets of Points*, p. 250, § 157.
‡ *Id.*, § 163, p. 256.

continuous function $\leqslant f(x)$. Also

$$\int_0^c [K+f(x)]\,dx = Kc+\int_0^c f(x)\,dx,$$

and this is, by the preceding article, the upper bound of the integrals of upper semi-continuous function $\leqslant K+f(x)$, that is, of $\int_0^c [K+g(x)]\,dx$, that is, $Kc+$upper bound of $\int_0^c g(x)\,dx$, that is, $Kc+$upper bound of the integrals of upper semi-continuous functions $\leqslant f(x)$.

Hence, *whatever bounded function $f(x)$ may be, its lower generalised integral is the upper bound of the integrals of upper semi-continuous functions $\leqslant f(x)$.*

Similarly its upper generalised integral is the lower bound of the integrals of lower semi-continuous functions $\geqslant f(x)$.

13. Let $f(x)$ be any bounded function defined at every point of the interval $0 \leqslant x \leqslant c$.

Let $g(x)$ and $h(x)$ be two upper semi-continuous functions $\leqslant f(x)$. Let us form a function $q(x)$ whose value is the greater of the values of $g(x)$ and $h(x)$ at each point x. Then $q(x) \leqslant f(x)$ at every point. Also $q(x)$ is upper semi-continuous (§ 3).

In this way, given any succession of upper semi-continuous functions less than $f(x)$, we can form a monotone ascending sequence of upper semi-continuous functions from them, each function being still $\leqslant f(x)$, and the unique limiting function of this monotone sequence will be not less than the upper function of the original succession, since each function of the monotone sequence is at every point not less than the corresponding function in the succession.

Let $I(c)$ denote the upper bound of the integrals (ordinary upper integrals) of all upper semi-continuous functions $\leqslant f(x)$, the limits of integration being 0 and c. Then we can find a succession of such functions whose upper integrals have $I(c)$ as unique limit. Forming from this succession a monotone ascending sequence of such functions, in the manner indicated above, the integrals of these will also form a monotone sequence, and each integral will be not less than the integral of the corresponding function in the succession. Hence the unique limit of the two sequences of integrals will be the same, viz., the upper bound $I(c)$ of all such integrals.

Thus we have proved that *we can find a monotone sequence*

$$a_1(x) \leqslant a_2(x) \leqslant \ldots \leqslant a_n(x) \leqslant \ldots,$$

of upper semi-continuous functions $\leqslant f(x)$, *whose integrals*

$$\int_0^c a_1(x)\,dx \leqslant \int_0^c a_2(x)\,dx \leqslant \ldots \leqslant \int_0^c a_n(x)\,dx \leqslant \ldots,$$

have as limit the upper bound $I(c)$ *of the integrals of all upper semi-continuous functions less than* $f(x)$.

14. Let $a(x)$ be the *lu*-function which is the limit of $a_n(x)$, when n is indefinitely increased; we have, of course,

$$a_n(x) \leqslant a(x) \leqslant f(x).$$

Also, by § 13,
$$\int_0^c a(x)\,dx = I(c),$$

since the integral of the limit of a monotone sequence is the limit of the integrals, if finite.

Hence, though the function $a(x)$ is not uniquely determined by the function $f(x)$ and the interval $(0, c)$; yet, however we vary the choice of the functions $a_n(x)$, subject to the conditions given, two such functions $a(x)$ will only differ at a set of content zero.

Conversely, of course, any *lu*-function $\leqslant f(x)$ whose generalised integral has the value $I(c)$ has all the characteristic properties of the function $a(x)$, viz., it is the limit of a monotone ascending sequence of upper semi-continuous functions $\leqslant f(x)$, such that the limit of their integrals is the lower generalised integral of $f(x)$ from 0 to c.

If we assume the theory of generalised integration already developed, we may now, by the preceding article, say that *the lower generalised integral of any bounded function* $f(x)$ *is the upper limit of the integrals of a monotone ascending sequence of upper semi-continuous functions* $\leqslant f(x)$.

If, however, we prefer not to assume that theory, we shall *define* $I(c)$ to be *the generalised lower integral* of $f(x)$, and the same statement may be made.

Thus, by definition, or as an immediate deduction from the above statement,

$$\operatorname*{Lt}_{n=\infty} \int_0^c a_n(x)\,dx = \int_0^c a(x)\,dx = \int_0^c f(x)\,dx.$$

15. Similarly, if $J(c)$ denote the lower bound of the integrals (ordinary lower integrals) of lower semi-continuous functions $\geqslant f(x)$, we can find a monotone descending sequence of lower semi-continuous functions $\geqslant f(x)$,

$$b_1(x) \geqslant b_2(x) \geqslant \ldots,$$

whose integrals have $J(c)$ as limit.

Hence, by definition, or otherwise, *the upper generalised integral of any bounded function $f(x)$ is the lower limit of the integrals of a monotone descending sequence of lower semi-continuous functions $\geqslant f(x)$.*

Denoting by $b(x)$ the *ul*-function which is the limit of $b_n(x)$, when n is indefinitely increased, we have, of course,

$$f(x) \leqslant b(x) \leqslant b_n(x),$$

$$\operatorname*{Lt}_{n=\infty} \int_0^c b_n(x)\,dx = \int_0^c b(x)\,dx = \overline{\int_0^c} f(x)\,dx.$$

16. Since

$$b_n(x) \geqslant b_{n+1}(x),$$

$$a_n(x) \leqslant a_{n+1}(x),$$

we have

$$b_n(x) - a_n(x) \geqslant b_{n+1}(x) - a_{n+1}(x);$$

also, since changing the sign of an upper semi-continuous function makes it a lower semi-continuous function, and the sum of two lower semi-continuous functions is a lower semi-continuous function, it follows that $b_n(x) - a_n(x)$ is a lower semi-continuous function.

Hence $b_1(x) - a_1(x) \geqslant b_2(x) - a_2(x) \geqslant \ldots \geqslant b_n(x) - a_n(x) \leqslant \ldots$

is a monotone descending sequence of lower semi-continuous functions. Moreover, each of these functions is $\geqslant 0$, since for every integer n,

$$a_n(x) \leqslant f(x) \leqslant b_n(x).$$

Now

$$\int_0^c [b_n(x) - a_n(x)]\,dx = \int_0^c b_n(x)\,dx + \int_0^c [-a_n(x)]\,dx,$$

since the (ordinary lower) integral of the sum of two lower semi-continuous functions is the sum of their integrals.*

But $$\underline{\int}_0^c [-a_n(x)]\,dx = -\overline{\int}_0 a_n(x)\,dx,$$

therefore $$\underline{\int}_0^c [b_n(x)-a_n(x)]\,dx = \underline{\int}_0^c b_n(x)\,dx - \overline{\int}_0^c a_n(x)\,dx.$$

Hence it follows that the lower integrals of the lower semi-continuous functions of the monotone descending sequence last written down, which, of course, form themselves a monotone descending sequence of positive quantities, have $J(c)-I(c)$ for their limit.

17. In the case of all bounded functions which are capable of mathematical expression
$$J(c) = I(c);$$
when this is the case, the function $f(x)$ is said to be *a summable function.* Now the points at which
$$b(x)-a(x) > k,$$
are points at which $$b_n(x)-a_n(x) > k,$$
for *all* values of n, since these latter functions form a monotone descending sequence with $b(x)-a(x)$ as limit.

Also, since $b_n(x)-a_n(x)$ is lower semi-continuous, the points at which it is $\leqslant k$ form a closed set, therefore the points at which it is $> k$ form a set of intervals. If the content of these intervals be $S_n(k)$, the (ordinary lower) integral of $b_n(x)-a_n(x)$ is accordingly not less than $kS_n(k)$. But this integral has zero as limit, therefore $S_n(k)$ has zero as limit. Hence the content of the inner limiting set of points at which
$$b(x)-a(x) > k$$
has zero as limit.† Thus $b(x)-a(x)$ is zero except at a set of points of content zero.

Since $f(x)$ lies between $a(x)$ and $b(x)$, it follows that, *when $f(x)$ is a summable function, $a(x)$, $f(x)$ and $b(x)$ differ only at a set of content zero.*

18. The functions $a(x)$ and $b(x)$ we formed with reference to the segment $(0, c)$, and are, of course, not uniquely determined by that

* W. H. Young, "On an Extension of the Heine-Borel Theorem," *Messenger of Math.*, January, 1904.

† Young's *Theory of Sets of Points*, p. 110, § 62.

segment and the function $f(x)$. The last result, however, shews that, however formed, provided $f(x)$ is summable, they only differ from $f(x)$ at a set of content zero in the segment $(0, c)$; and therefore in any smaller segment only differ from $f(x)$ at a set of content zero.

Hence $\int_0^x b_n(x)\,dx$ and $\int_0^x a_n(x)\,dx$

for fixed x, have, as n increases indefinitely, the same limit

$$I(x) = J(x),$$

which is at the same time the upper bound of the upper integrals of upper semi-continuous functions less than $f(x)$ $\left(\text{since such integrals never exceed } \int_0^x b_n(x)\,dx\right)$, and the lower bound of the lower integrals of lower semi-continuous functions greater than $f(x)$ $\left(\text{since such integrals never are less } \int_0^x a_n(x)\,dx\right)$.

Such a function $a(x)$ is called a bounding *lu*-function of $f(x)$, and such a function $b(x)$ is called a bounding *ul*-function of $f(x)$.

Conversely, it is obvious that any *lu*-function $\leqslant f(x)$ [*ul*-function $\geqslant f(x)$], which differs from $f(x)$ only at a set of content zero is a bounding lower-upper (upper-lower) function of $f(x)$, provided $f(x)$ is a bounded summable function.

19. THEOREM.—*If $f(x)$ and $F(x)$ are bounded summable functions, and $a(x)$ and $A(x)$ are corresponding bounding lu-functions, and $b(x)$ and $B(x)$ corresponding bounding ul-functions, then if*

$$f(x) \leqslant F(x),$$

we may assume that

$$a(x) \leqslant A(x) \quad \text{and} \quad b(x) \leqslant B(x).$$

For, if the monotone ascending sequences of upper semi-continuous functions whose limits are $a(x)$ and $A(x)$ respectively, be

$$a_1(x) \leqslant a_2(x) \leqslant \ldots,$$

and

$$A_1(x) \leqslant A_2(x) \leqslant \ldots,$$

we may form a new monotone ascending sequence of functions

$$g_1(x) \leqslant g_2(x) \leqslant \ldots,$$

such that $g_n(x)$ has as value the greater of the two values $a_n(x)$ and $A_n(x)$.

By § 3, $g_n(x)$ is upper semi-continuous. Also at any point at which $g_n(x)$ differs from $A_n(x)$, it is $= a_n(x)$, and therefore $\leqslant f(x)$ and therefore $\leqslant F(x)$. Also since

$$A_n(x) \leqslant g_n(x);$$

and therefore

$$\int A_n(x)\,dx \leqslant \int g_n(x),$$

we must have

$$\operatorname*{Lt}_{n=\infty} \int A_n(x)\,dx = \operatorname*{Lt}_{n=\infty} \int g_n(x),$$

since the limit on the left is the highest possible.

Thus the sequence of functions g_n may be taken instead of the sequence A_n, in which case we have

$$a_n(x) \leqslant A_n(x);$$

and therefore

$$a(x) \leqslant A(x).$$

This proves the first result.

Similarly, if the sequences defining $b(x)$ and $B(x)$ are

$$b_1(x) \geqslant b_2(x) \geqslant \ldots,$$

$$B_1(x) \geqslant B_2(x) \geqslant \ldots.$$

We may so modify the first sequence, by changing $b_n(x)$ into $B_n(x)$ at any point where $B_n(x)$ was less than $b_n(x)$, that

$$b_n(x) \leqslant B_n(x).$$

This change, by § 3, does not alter the lower semi-continuity of $b_n(x)$, nor the monotony of the first sequence.

Also at any point at which the value of $b_n(x)$ has been altered it is become $\geqslant F(x)$, and therefore certainly $\geqslant f(x)$. Also since the value of $b_n(x)$, where modified, has been reduced, the limit of $\int b_n(x)\,dx$, already as small as possible, has been unaltered. Thus the modification is allowable and it makes

$$b(x) \leqslant B(x).$$

This proves the theorem.

20. *Construction of Limiting Upper-Lower and Lower-Upper Functions of a Bounded Measurable Function.*

Let K be the lower and K' the upper bound of the given bounded function $f(x)$. Divide the interval (K, K') into n equal parts, and let e_n be the length of each, and, for shortness, write

$$k_r = K + re_n.$$

Let I_r be the content of the set of points at which $f(x) \geqslant k_r$; these points form a measurable set, if, as we assume, f is a measurable function. Consequently we can find a closed sub-set of content greater than $I_r - \frac{1}{n.2^n}$. Let such a set be denoted by I'_r, for all values of r from 0 to $n-1$, k_0 being understood to mean K.

Let the function $a_n(x)$ be defined as follows :—

$$\begin{aligned} a_n(x) &= k_{n-1} \text{ at every point of } I'_{n-1} \\ &= k_{n-2} \text{ at every point of } I'_{n-2} - I'_{n-1} \\ &= k_r \text{ at every point of } I'_r - I'_{r+1} \ (1 \leqslant r \leqslant n-2) \\ &= K \text{ at all the remaining points.} \end{aligned}$$

Then $a_n(x)$ is an upper semi-continuous function $\leqslant f(x)$, and the functions form a monotone ascending sequence,

$$a_1(x) \leqslant a_2(x) \leqslant \dots .$$

Again, let the function $b_n(x)$ be defined as follows :—

$$\begin{aligned} b_n(x) &= k_1 \text{ at all the points of } I'_0 - I'_1 \\ &= k_2 \text{ at all the points of } I'_1 - I'_2 \\ &= k_{r+1} \text{ at all the points of } I'_r - I'_{r+1} \ (0 \leqslant r \leqslant n-2) \\ &= K' \text{ at all the remaining points.} \end{aligned}$$

Then $b_n(x)$ is a lower semi-continuous function $\geqslant f(x)$, and the functions form a monotone descending sequence,

$$b_1(x) \geqslant b_2(x) \geqslant \dots .$$

Moreover $b_n(x) - a_n(x) = e_n$, except at the points of a set of content $< \frac{1}{2^n}$, at which it has the value $K' - K$.

Thus the functions $b_n(x) - a_n(x)$ form a monotone descending sequence of functions with zero as unique limit, except at a set of points which, being the generalised inner limiting set of measurable sets whose contents have zero as limit, is a set of zero content.

Hence, denoting the *lu*-function which is the limit of the functions $a_n(x)$ by $a(x)$, and the *ul*-function which is the limit of the function $b_n(x)$ by $b(x)$, it follows that $b(x)$ differs from $a(x)$; and therefore both differ from $f(x)$, which always lies between them, only at a set of content zero.

Hence also the unique limit of

$$\int_0^c b_n(x) - \int_0^c a_n(x) = \int_0^c [b_n(x) - a_n(x)]\, dx$$

is zero, so that the integral of $b_n(x)$ and the integral of $a_n(x)$ have the same limit. Hence, by § 17, $f(x)$ is a summable function, and this common limit is the common value of the integrals of $f(x)$, $a(x)$, and $b(x)$.

Also $a(x)$ and $b(x)$, so constructed, may be taken to be the bounding *lu*- and *ul*-functions of the bounded measurable function $f(x)$.

Thus *any bounded measurable function is summable*,* and in particular this is the case with bounded *lu*'s and *ul*'s, *ulu*'s and *lul*'s, &c.

21. If the function $f(x)$ is a monotone increasing (decreasing) function, the semi-continuous functions $a_n(x)$ and $b_n(x)$, constructed as in the preceding article, will not be monotone, but a very slight modification in the construction renders them monotone.

In fact, the points at which $f(x) \geqslant k_n$ form an interval, closed on the right (left) but not necessarily on the left (right). If the interval is closed, we take it for the set I_n'. If, however, it does not include its left (right) hand end-point P, we take for I_n' a closed interval got by subtracting from the whole interval a small segment at the end in question. In this small segment, instead of, as in the preceding article, assigning to a_n as value the lower bound of $f(x)$, we assign the value $f(P)$, and this value we also retain at P. In this way the function $a_n(x)$ is constant in a finite number of intervals and monotone increasing (decreasing).

The same device renders b_n monotone increasing (decreasing).

It then follows, of course, that $a(x)$ and $b(x)$ are monotone in the same sense as $f(x)$.

22. So far we have confined our attention to bounded functions, and, in point of fact, important modifications are necessary, when we come to deal with unbounded functions. Two courses are possible. We may either proceed precisely in the case of such functions as in the theory of Riemann integrals, and employ De la Vallée Poussin's definition for absolutely convergent improper integrals, and Harnack's definition for non-

* H. Lebesgue, *Leçons sur l'Intégration*, Gauthier-Villars, 1904, p. 115. It should be noticed that the use of the words *measurable* and *summable* here is that adopted at present by Lebesgue. The definition of *summable* (= measurable) in Hobson's *Functions of a Real Variable*, p. 391, is in accordance with Lebesgue's earlier usage, since abandoned.

absolutely convergent integrals. In this way we still preserve, in the case of the absolutely convergent integrals all the advantages afforded by the use of monotone sequences. We may, however, in the case of such integrals also employ with convenience the slight modification of the method already explained for bounded functions.

Take first a positive function. Then, if it is unbounded above, we shall no longer necessarily be able to obtain a descending sequence of lower semi-continuous functions satisfying our requirements.

We can, however, make use of the ascending sequence of upper semi-continuous functions, provided the integral, or the lower integral, in the generalised sense, of $f(x)$ is finite.

For, if in accordance with the method of De la Vallée Poussin, we form the auxiliary sequence of bounded functions

$$f_1(x) \leqslant f_2(x) \leqslant \ldots \leqslant f_m(x) \leqslant \ldots,$$

where $f_m(x) = f(x),$

wherever $f(x) \leqslant m,$

and at other points $f_m(x) = m,$

we have, by the definition of the improper lower generalised integral,

$$\underline{\int} f(x)\,dx = \operatorname*{Lt}_{m=\infty} \int f_m(x)\,dx$$

$$= \operatorname*{Lt}_{m=\infty} \int a_m(x)\,dx,$$

where $a_m(x)$ is a bounding *lu*-function of $f_m(x)$.

Since the functions $f_m(x)$ form a monotone ascending sequence, we may take it, by § 19, that the functions $a_m(x)$ do so also. Thus, if $a(x)$ is the limit of $a_m(x)$, $a(x)$ is a *llu*, that is, a *lu*-function.

Also, since the integrals form a monotone ascending sequence with a finite limit

$$\int a(x)\,dx = \operatorname*{Lt}_{m=\infty} \int a_m(x)\,dx = \underline{\int} f(x)\,dx.$$

Thus the lower generalised integral of a positive function, and therefore of any function which is bounded below, is, if finite, the generalised integral of a *lu*-function, nowhere greater than the function.

Similarly, the upper generalised integral of a function which is bounded above, is, if finite, the generalised integral, of an *ul*-function, nowhere less than the function.

In symbols, *if $f(x)$ is bounded below and has a finite generalised lower integral,*

$$\underline{\int} f(x)\,dx = \int a(x)\,dx = \dot{\int} f(x)\,dx,$$

if f is summable; while, if $f(x)$ is bounded above and has a finite generalised upper integral,

$$\overline{\int} f(x)\,dx = \int b(x)\,dx = \dot{\int} f(x)\,dx,$$

if f is summable; where $a(x)$ is a lu, and $b(x)$ an ul, the former $\leqslant f(x)$, and the latter $\geqslant f(x)$.

If the function is unbounded both above and below, and possesses an absolutely convergent improper integral, our method will give us the value of this integral as the difference of the integrals of two *lu*-functions, or, which is the same thing, the function which replaces f for purposes of integration is the sum of a *lu* and an *ul*.

Such a function is plainly both an *ulu* and a *lul* function, so that we can, in two distinct ways, apply the method of monotone sequences, to reduce the consideration of such functions to that of bounded continuous functions.

In this connection we may remark in addition that, if the nature of the argument is such as to permit of it, we may by the use of a further sequence, not monotone but uniform, secure that these continuous functions have a differential coefficient everywhere, in virtue, for example, of Weierstrass's theorem that a continuous function may be expressed as the limit of polynomials.

23. THEOREM.—*If $f(x)$ and $F(x)$ are bounded measurable functions, whose bounding functions are respectively $a(x)$, $b(x)$ and $A(x)$, $B(x)$, the bounding functions of $f(x)+F(x)$ are $a(x)+A(x)$ and $b(x)+B(x)$, and those of $f(x)\,F(x)$ are $a(x)\,A(x)$ and $b(x)\,B(x)$, provided in the latter case $f(x)$ and $F(x)$ are both positive.*

For, by § 7, $a(x)+A(x)$ and $a(x)\,A(x)$ are *lu*-functions, and the former is $\leqslant f(x)+F(x)$, but differs from it only at a set of content zero, while the same is true of the latter function, and $f(x)\,F(x)$. This therefore proves the theorem for the a's, by § 18.

The proof for the b's is precisely similar.

COR.—*If $f(x)$ and $F(x)$ are bounded below and summable, the above theorem holds for the a's.*

If they are bounded above, the above theorem holds for the b's.

24. We now proceed to give some examples of the use of the method in special cases. The first is an inequality, known in its original form as Schwarz's* inequality.

THEOREM.—*If f and F are summable functions, bounded or unbounded,*

$$\left(\int_0^c fF\,dx\right)^2 \leqslant \int_0^c f^2 dx \int_0^c F^2 dx.$$

It is evidently only necessary to prove the theorem for positive functions $f(x)$ and $F(x)$.

CASE 1.—*Let f and F be continuous functions.*

In this case the left-hand side of the inequality to be proved is the double integral

$$\int_0^c \int_0^c f(x)\,F(x)\,f(y)\,F(y)\,dx\,dy,$$

and the right-hand side is the double integral

$$\int_0^c \int_0^c [f(x)\,F(y)]^2\,(dx\,dy) = \int_0^c \int_0^c [f(y)\,F(x)]^2\,(dx\,dy)$$

$$= \tfrac{1}{2}\int_0^c \int_0^c \{[f(x)\,F(y)]^2 + [f(y)\,F(x)]^2\}\,(dx\,dy),$$

whence the theorem is obvious.

CASE 2.—*Let f and F be both lower or both upper semi-continuous functions.*

Let $$a_1 \leqslant a_2 \leqslant \ldots$$

and $$A_1 \leqslant A_2 \leqslant \ldots$$

be monotone increasing sequences of continuous functions having respectively the lower semi-continuous functions f and F as limit.

Then $$a_1 A_1 \leqslant a_2 A_2 \leqslant \ldots$$

is a monotone increasing sequence of continuous functions with fF as limit. Also

$$\int_0^c fF\,dx = \operatorname*{Lt}_{n=\infty} \int_0^c a_n A_n dx.$$

* Quoted in Lebesgue's paper, "Sur les intégrales singulières," 1910, *Annales de Toulouse*, p. 37 *seq.*

Similarly,

$$\int_0^c f^2 dx = \operatorname*{Lt}_{n=\infty} \int_0^c a_n^2 dx,$$

$$\int_0^c F^2 dx = \operatorname*{Lt}_{n=\infty} \int_0^c A_n^2 dx.$$

But, by the first case,

$$\left(\int_0^c a_n A_n dx\right)^2 \leqslant \int_0^c a_n^2 dx \int_0^c A_n^2 dx.$$

Hence, proceeding to the limit with n, the theorem follows in this case. The proof is similar when f and F are upper semi-continuous functions.

CASE 3.—*Let f and F be both lu's (or both ul's).*

The proof is formally the same as in Case 2, the functions $a_1, a_2, \ldots$, $A_1, A_2, \ldots$ being now upper (or lower) semi-continuous functions.

This case includes that where f and F are any bounded measurable functions, or where both are bounded above or both are bounded below, using the results of § 22, by which the integrals of such functions are replaced by the integrals of *lu*'s or of *ul*'s.

CASE 4.—*Let f and F be both ulu's (or both lul's).*

The proof is formally the same as in Case 2, the functions $a_1, a_2, \ldots$, $A_1, A_2, \ldots$ being now *lu*'s (or *ul*'s).

This case includes every possible case, since, as shewn in § 22, the integral of any measurable function is the integral of either of the types here considered.

Thus in every case the theorem holds.

25. The inequality

$$\left|\int_0^c 2fF\,dx\right| \leqslant \int_0^c |2fF|\,dx \leqslant \int_0^c f^2 dx + \int_0^c F^2 dx$$

may be proved, of course, in a similar manner, but still more easily.

26. The next application of the method is to term-by-term integration.

THEOREM.—*If $f_1(x), f_2(x), \ldots$ be any sequence of positive functions, each $\leqslant V(x)$, where $V(x)$ is a summable function, the sequence having zero as unique limit, then term-by-term integration is allowable.*

For let $g_n(x)$ be the function whose value at any point x is the upper bound of the values of $f_n(x), f_{n+1}(x), \ldots$ at that point, then

$$0 \leqslant f_n(x) \leqslant g_n(x) \leqslant V(x).$$

Therefore $g_n(x)$ is summable, and

$$0 \leqslant \int f_n(x)\,dx \leqslant \int g_n(x)\,dx \leqslant \int V(x)\,dx.$$

But the functions $g_n(x)$ clearly form a monotone descending sequence with zero as limit; hence

$$\operatorname*{Lt}_{n=\infty} \int g_n(x)\,dx = 0,$$

whence, by the preceding inequality,

$$\operatorname*{Lt}_{n=\infty} \int f_n(x)\,dx = 0,$$

which proves the theorem.

Hence we have as an immediate corollary a theorem given by Lebesgue* and included in one due to Vitali.

COR.—*If $f_1(x), f_2(x), \ldots$ be any sequence of summable functions whose unique limiting function $f(x)$ is summable, we may integrate term-by-term, provided for all values of n and x,*

$$|f(x)-f_n(x)| \leqslant V(x),$$

where $V(x)$ is a summable function.

26. Hitherto we have confined ourselves to a single variable, but the whole theory applies, of course, when x is the ensemble of two or more variables. In this case we have theorems involving multiple and repeated integrals, as well as parametric integrals. We shall consider next some results of parametric integration.

If $f(x, y)$ is an upper (lower) semi-continuous function having a finite upper (lower) bound, it is the limit of a monotone descending (ascending)

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* H. Lebesgue, "Sur la méthode de M. Goursat pour la résolution de l'équation de Fredholme," *Bull. de la Soc. Math. de France*, 1908. "Sur les intégrales singulières," 1910, p. 50, where reference is made to Vitali.

sequence of continuous (bounded) functions.* Hence $\int f(x, y)\,dx$, which is the limit of the similar monotone sequence of the integrals of those continuous functions, is *also an upper (lower) semi-continuous function.*

If, however, we omit the condition as to the finitude of the upper (lower) bound, the result is different; in this case the parametric integral is a lu-function (ul-function).

For, if $f(x, y)$ is an upper (lower) semi-continuous function of (x, y), unbounded above (below),

$$f(x, y) = \underset{M=+\infty\ (\text{or}\ -\infty)}{\text{Lt}} f(x, y, M),$$

where $f(x, y, M)$ is, by § 8, a function of the same type as $f(x, y)$, but is bounded above (below). Since the functions $f(x, y, M)$ form a monotone sequence, $\int f(x, y)\,dx$ is the limit of the monotone sequence of their integrals, and is therefore the limit of a monotone ascending (descending) sequence of upper (lower) semi-continuous functions, as has just been pointed out. This proves the statement made above.

Hence, in particular, if $f(x, y)$ is an unbounded continuous function of (x, y), $\int f(x, y)\,dx$ is, by § 10, a function of Baire's first class.

Similarly, *if $f(x, y)$ is a lu (ul) function, its parametric integral is, if the upper (lower) bound be finite, also a lu (ul). But, if the upper (lower) bound be infinite, it is an ulu (lul).*

The series of these theorems may be extended indefinitely.

27. THEOREM.—*If u is a bounded function of (x, y) which is a continuous, or an upper or lower semi-continuous function of (x, y), or if it is a lu- or an ul-function of (x, y), ..., and v is a not necessarily bounded but positive function of (x, y) having an integral $\int_p^q v\,dx$ with respect to x which is a continuous function of y, then $\int_p^q uv\,dx$ exists, and is a function of y of the same type as u of (x, y).*

The case when u is a continuous function of (x, y) is included in the more general case treated in a recent paper of mine,† when the discontinuities of u with respect to (x, y) form for each fixed value of y a set of

* "On Monotone Sequences of Continuous Functions," p. 526.

† "On Parametric Integration," *Monatshefte f. Math. und Physik*, Bd. XXI (1910), p. 143, where also the method of monotone sequences is employed.

zero content. Assuming this proved, the other cases may be proved by monotone sequences, since, u being bounded, all the integrals which occur are finite.

For, if $u(x, y)$ is the limit of a monotone ascending (descending) sequence of functions of (x, y), $u_1(x, y)$, $u_2(x, y)$, ..., for each of which the theorem is true, so that $\int_p^q u_n v\,dx$ is a function of y of the same type as u_n is of (x, y), then since the functions $u_1 v$, $u_2 v$, ... are the products of the functions of the former sequence with an essentially positive function, so that they, and therefore their integrals, form a sequence monotone in the same sense as before, it follows that the limit of the sequence of integrals, that is, $\int_p^q uv\,dx$ is a function of y of the same type as u is of (x, y).

28. THEOREM.—*If $u(x)$ is any bounded function of x, and $v(x, y)$ is a not necessarily bounded but essentially positive function of (x, y) having an integral $\int_p^q v\,dx$ with respect to x which is a continuous function of y, then $\int_p^q uv\,dx$ exists, and is a function of Baire's first class.*

For, if $a(x)$ and $b(x)$ denote bounding *lu*- and *ul*-functions of $u(x)$, they are respectively *lu*- and *ul*-functions with respect to the ensemble (x, y), since they do not involve y.

Hence, by the preceding theorem,

$$\int_p^q a(x)\, v(x, y)\,dx \text{ is a } lu\text{-function of } y,$$

and

$$\int_p^q b(x)\, v(x, y)\,dx \text{ is an } ul\text{-function of } y.$$

But, since $a(x)$, $b(x)$ and $u(x)$ differ only at a set of values of x of content zero, it follows that on each ordinate av, bv, and uv differ only at a set of values of x of content zero, so that their integrals with respect to x, two of which have been shewn to exist, and to be respectively a *lu* and an *ul*, are equal, and each is both a *lu*- and also an *ul*-function of y, that is, by § 10, it is a function of Baire's first class.

29. The former of the two theorems of the preceding article is a simpler form of the following more general theorem, the proof of which

is in all essentials the same as in the simpler case, Theorem 6, p. 145, of the paper quoted being used, instead of Theorem 5, p. 143.

THEOREM.—*If u is a bounded function of (x, y) which is lower semi-continuous with respect to (x, y), or is a lu, or a lul, ..., with respect to (x, y), and v is a not necessarily bounded but essentially positive function of (x, y) having an integral $\int_p^q v\,dx$ with respect to x, which is a lower semi-continuous function of y, then $\int_p^q uv\,dx$ exists, and is a function of y of the same type as u is of (x, y).*

If instead of being lower semi-continuous $\int_p^q v\,dx$ is upper semi-continuous with respect to y, then $\int_p^q uv\,dx$ exists, and is a function of y of the next type after the type to which u belongs.

Similarly interchanging "upper" and "lower," we have the alternative theorem.

29. We now proceed to discuss the inequalities* between the generalised double and repeated integrals of a function of two variables.

For continuous functions of the ensemble (x, y) we know that the double and repeated integrals are equal. Hence it at once follows, by the method of monotone sequences that *the same is true for any bounded semi-continuous functions, and for any bounded lu-functions or ul-functions, and for any bounded lul-functions or ulu-functions, and so on.*

For, if $$a_1(x, y) \leqslant a_2(x, y) \leqslant \dots$$

is a monotone ascending sequence of functions for which the theorem is true,

$$\int_0^x a_1(x, y)\,dx \leqslant \int_0^x a_2(x, y)\,dx \leqslant \dots,$$

and $$\int_0^x\int_0^y a_1(x, y)(dx\,dy) \leqslant \int_0^x\int_0^y a_2(x, y)(dx\,dy) \leqslant \dots,$$

are also monotone ascending sequences, so that if the limit of the first

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* W. H. Young, "On the Inequalities connecting the Double and Repeated Upper and Lower Integrals of a Function of Two Variables" (1908), *Proc. London Math. Soc.*, Ser. 2, Vol. 6, pp. 240-254

sequence is $a(x, y)$, that of the second sequence is $\int_0^x a(x, y)\,dx$, and that of the third sequence is $\int_0^x \int_0^y a(x, y)(dx\,dy)$.

Integrating the second sequence with respect to y, we get, by hypothesis, the third sequence. Also since the second sequence is monotone, the limit of the integral so obtained is

$$\int_0^y dy \int_0^x a(x, y)\,dx,$$

which is accordingly equal to the double integral of $a(x, y)$. Similarly the other repeated integral has the same value.

This proves the equality of the double and repeated integrals for a lower semi-continuous function. Similarly it may be proved for an upper semi-continuous function. The proof given then shews that it is true for a *lu*-function, and so on.

30. Next, let $f(x, y)$ be any bounded function. Let $a(x, y)$ be a bounding *lu*-function of $f(x, y)$ with respect to (x, y). Then since

$$a(x, y) \leqslant f(x, y),$$

and $a(x, y)$ is, for fixed x, a *lu*-function with respect to y, we may take it that

$$a(x, y) \leqslant A(x, y),$$

where $A(x, y)$ is, for fixed x, a bounding *lu*-function of $f(x, y)$. Hence

$$\int a(x, y)\,dy \leqslant \int A(x, y)\,dy \leqslant \underline{\int} f(x, y)\,dy.$$

Taking integrals with respect to x on each side,

$$\int dx \int a(x, y)\,dy \leqslant \underline{\int} dx \underline{\int} f(x, y)\,dy.$$

But the left-hand side is equal to the double integral of $a(x, y)$, since $a(x, y)$ is a *lu*-function with respect to (x, y), and is therefore equal to the lower double integral of f. Thus we have the inequality

$$\underline{\int}\underline{\int} f(x, y)(dx\,dy) \leqslant \underline{\int} dx \underline{\int} f(x, y)\,dy,$$

that is, lower double $\leqslant$ lower-lower.

Similarly, upper-upper $\leqslant$ upper double.

But obviously lower-lower $\leqslant$ upper-upper,

so that *for any bounded functions whatever, the integrals being generalised integrals,*

$$\textit{lower double} \leqslant \textit{lower-lower} \leqslant \textit{upper-upper} \leqslant \textit{upper double},$$

just as for ordinary Riemann integrals.

The known result at once follows that, if the double integral exists, the repeated integrals exist and are equal to it, whether or not $f(x, y)$ be bounded.*

31. Next let $f(x, y)$ be bounded above.

Then, as shown in § 22, we may still replace it by a *lu*-function, and write

$$\underline{\int}\underline{\int} f(x, y)(dx\,dy) = \iint a(x, y)(dx\,dy).$$

Hence, precisely as in the preceding article, we have

$$\textit{lower double} \leqslant \textit{lower-lower},$$

provided $f(x, y)$ is bounded above.

Similarly, $$\textit{upper-upper} \leqslant \textit{upper double},$$

provided the function is bounded below.

32. Returning to the case of a single variable, I give a proof of the Second Theorem of the Mean, which may be compared with those recently given by Hobson and Lebesgue.†

THEOREM (*Second Theorem of the Mean*).—*If $f(x)$ is a positive monotone decreasing function and $F(x)$ is a function having a generalised integral, such that*

$$n \leqslant \int_0^x F(x)\,dx \leqslant N,$$

then‡

$$nf(0) \leqslant \int_0^x f(x)\,F(x)\,dx \leqslant Nf(0).$$

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* E. W. Hobson, "On some Fundamental Properties of Lebesgue Integrals in a Two-Dimensional Domain," 1909, *Proc. London Math. Soc.*, Ser. 2. Vol. 8, pp. 22–40, where further references may be found.

† E. W. Hobson, "On the Second Mean-Value Theorem of the Integral Calculus," 1908, *Proc. London Math. Soc.*, Ser. 2, Vol. 7, pp. 14–24; H. Lebesgue, *loc. cit.*, § 23, *supra.*

‡ If $f(x)$ is discontinuous at $x = 0$, we must understand by $f(0)$ the limit, necessarily unique, of the values of $f(x)$ in the right-hand neighbourhood of the zero point.

CASE 1.—*Let $F(x)$ be any continuous function, and $f(x)$ a continuous function having a right-hand differential coefficient $f'(x)$ at every point which is continuous at all but a finite number of points.*

Then, if we denote the (ordinary) integral of $F(x)$ by $G(x)$, $G(x)$ has at every point a differential coefficient $= F(x)$. Hence $f(x)\,G(x)$ has at every point a right-hand differential coefficient, viz.,

$$f'G+fF.$$

Now this is continuous at all but a finite number of points, and is therefore integrable, so that, by a well-known theorem in the theory of ordinary, or Riemann, integration, its integral is $f(x)\,G(x)$. Hence integrating by parts,

$$\int_0^x f(x)\,F(x)\,dx = [f(x)\,G(x)]_0^x - \int_0^x f'(x)\,G(x)\,dx$$

$$= f(x)\,G(x) + \int_0^x [-f'(x)\,G(x)]\,dx.$$

Hence, $f(x)$ and $-f'(x)$ being $\geqslant 0$, we decrease (increase) the value of the right-hand side if we replace $G(x)$ by a lesser (greater) quantity. Replacing $G(x)$ first by n, we get on the right $nf(0)$, and then by N, we get $Nf(0)$, whence the required inequality follows.

CASE 2.—*Let $F(x)$ and $f(x)$ be any bounded lower (upper) semi-continuous functions.*

Then, as pointed out in § 3, they are respectively the limits of two monotone increasing (decreasing) sequences of continuous functions, having at every point right-hand differential coefficients, which are continuous except at a finite number of points. Moreover, if these sequences are

$$f_1(x),\ f_2(x),\ \ldots,\ f_n(x),\ \ldots,$$

and

$$F_1(x),\ F_2(x),\ \ldots,\ F_n(x),\ \ldots,$$

we may, from the method of construction given, assume that the functions $f_n(x)$, like $f(x)$ are monotone decreasing and essentially positive.

Hence, by the first case,

$$n_r f_r(0) \leqslant \int_0^x f_r(x)\,F_r(x)\,dx \leqslant N_r f_r(0),$$

where

$$n_r \leqslant \int_0^x F_r(x)\,dx \leqslant N_r.$$

Now, since the integrals $\int_0^x F_r(x)$ form a monotone sequence of continuous functions, we have, by the Theorem of the Bounds,* if n and n_r are the respective lower bounds, and N and N_r the upper bounds,

$$n = \underset{r=\infty}{\text{Lt}}\, n_r, \quad N = \underset{r=\infty}{\text{Lt}}\, N_r,$$

so that, proceeding to the limit,

$$nf(0) \leqslant \int_0^x f(x)\, F(x) \leqslant Nf(0).$$

Hence, if n was less than the lower bound, or N greater than the upper bound of the integral in question, the same inequality holds. Thus the theorem holds in this case.

CASE 3.—*Let $F(x)$ and $f(x)$ be bounded lu-functions (ul-functions).*

Then the proof given in Case 2, together with the result there proved, shews that the theorem is still true.

Similarly it is then seen to be true for bounded *lul*'s or *ulu*'s.

CASE 4.—*Let $F(x)$ be any bounded function.*

The function $f(x)$ is necessarily bounded, since it is $\geqslant 0$ and monotone decreasing, and $f(0)$ is, of course, supposed to be finite.

Let $a(x)$ and $A(x)$ be bounding lu-functions, and $b(x)$ and $B(x)$ bounding *ul*-functions of $f(x)$ and $F(x)$ respectively. Then $a(x)\,A(x)$ and $b(x)\,B(x)$ only differ from $f(x)\,F(x)$ at a set of content zero, so that

$$\int f(x)\, F(x)\, dx = \int a(x)\, A(x) dx = \int b(x)\, B(x) dx$$

Also, by Case 3,

$$\int_0^x a(x)\, A(x)\, dx \leqslant Na(0) \leqslant Nf(0),$$

and

$$nf(0) \leqslant nb(0) \leqslant \int_0^x b(x)\, B(x) dx.$$

From these three relations the result at once follows.

CASE 5.—*Let $F(x)$ be unbounded.*

First let $F(x)$ be unbounded above. Then it is the limit of a monotone ascending sequence of bounded functions $F_1(x)$, $F_2(x)$, ... ; and therefore,

* Theorem 1 of the paper quoted above on p. 18.

$f(x)$ being positive, and bounded, $f(x)\,F(x)$ is the limit of the monotone ascending sequence of bounded functions $f(x)\,F_r(x)$. Also since $F(x)$ is summable, so is $f(x)\,F(x)$, and its integral is the limit of that of $f(x)\,F_r(x)$.

But, by Case 4,

$$n_r f(0) \leqslant \int_0^x f(x)\,F_r(x)\,dx \leqslant N_r f(0),$$

whence, as in Case 2, the required result follows.

Similarly, if $F(x)$ is bounded above but unbounded below, the result follows.

Finally, if $F(x)$ is unbounded both above and below, it is the limit of a monotone ascending sequence of functions which are bounded above and unbounded below, so that, applying what has just been proved, and using the same argument, the result again follows.

Thus in every case the theorem is true.

33. The proof just given may not seem appreciably shorter, if indeed it is as short, as that, for example, given by Lebesgue. This is due, however, partly to the fact that I have been at pains to indicate precisely the nature of each step taken; it would be quite possible to so word the proof just given as to condense it into a few lines. In fact, as soon as the method employed is understood the result becomes intuitive. We may remark, however, that we may, if we please, employ a more convenient sequence of continuous functions than that made use of above, in virtue of the following theorem.

Every bounded lower (upper) semi-continuous function is expressible as the limit of a monotone ascending (descending) sequence of continuous functions, possessing at every point differential coefficients. We may, indeed, secure that the continuous functions in question possess each an n-th differential coefficient and without any alteration of the method of construction, that they are expandible in the whole interval in Taylor's series.

To prove this, we remark that the continuous functions, obtained as explained in § 3 in Baire's way, are represented by polygons of a finite number of sides. Taking any one of these functions, we may clearly approach it by means of functions represented by polygons with rounded corners, and it is plain that the approach may be chosen to be monotone in the same direction as before. From these new functions we can then,

as in § 4, choose out a new sequence, monotone as before, having the original semi-continuous function as limit.

COR.—*If the semi-continuous function is monotone, the continuous functions are also monotone, or may obviously be so chosen, without in any way altering the reasoning.*

Self-evident as this result now appears, it is clear that it removes almost every conceivable difficulty of exposition. The method of monotone sequences reduces, in fact, the study of the theory of integration to the case in which the functions with which we are concerned possess all the properties of the simple functions which the beginner encounters when he first comes to the calculus. Indeed nothing more is theoretically required than the consideration of the functions $x^m y^n$, ... In fact, wherever the passage by means of monotone sequences is allowed, every integration theorem satisfied by these simple functions will be satisfied by all summable functions.

34. THEOREM (of Integration by Substitution).*—*If $f(x)$ is a summable function, bounded or unbounded, and*

$$x = g(t) = \int_0^t h(t)\,dt,$$

be a monotone increasing function of t, whose value is k when $t = c$, then

$$\int_0^k f(x)\,dx = \int_0^c f[g(t)]\,h(t)\,dt.$$

We note that we may, if we please, suppose $h(t)$ to be a function which is positive, for it can at most differ from such a function at a set of points of zero content, since its integral is a monotone increasing function.

We remark at once that it will be sufficient to prove the theorem when $f(x)$ is a positive power of x, for it will then be obviously true for any polynomial. It will then follow, from Weierstrass's theorem, by which any continuous function can be expressed as the uniform limit of a

* Cp. Lebesgue's memoir already quoted "Sur les intégrales singulières," and E. W. Hobson, "On Change of Variable in a Lebesgue Integral," 1909, *Proc. London Math. Soc.*, Ser. 2, Vol. 8, pp. 10–22.

sequence of polynomials, that it is also true for any continuous function. In fact, if f and f_n be continuous functions, such that f_n converges uniformly to f as n increases indefinitely, we have

(1) $\int_0^c h(t)\,dt$ is independent of n and finite;

(2) $h(t)$ is a positive function;

(3) $f_n[g(t)]$ is a continuous* function of (n, t), where ∞ is included as one of the values of n;

$\int_0^c f_n[g(t)]\,h(t)\,dt$ is therefore a continuous function of n at the point $n = \infty$. In other words,

$$\operatorname*{Lt}_{n=\infty} \int_0^c f_n[g(t)]\,h(t)\,dt = \int_0^c f[g(t)]\,h(t)\,dt.$$

Plainly, also $$\operatorname*{Lt}_{n=\infty} \int_0^k f_n(x)\,dx = \int_0^k f(x)\,dx.$$

This proves the theorem for any continuous function f, provided it has been proved when f is a positive integral power of x.

It follows at once, by the method of monotone sequences, since h has been made a positive function of t, that the theorem is true for any positive bounded function f, and therefore for any bounded function.

A further application of the method of monotone sequences plainly allows us to state the theorem for any summable function f, bounded or unbounded. For, if one side of the equation to be proved is finite, as is, by hypothesis, the case with the left-hand side, the method plainly ensures that all the integrals obtained in proceeding to the limit on the right-hand side are also finite.

35. It is therefore sufficient to prove the theorem for the case in which $f(x) = x^m$.

* The variable n may, if we please, be made continuous in the usual manner. The theorem used is that quoted above from the *Monatshefte*.

Take the case $m = 1$. That is,

$$f(x) = x.$$

Here the right-hand side of the equality to be proved is

$$\begin{aligned}\int_0^c g(t)\,h(t)\,dt &= \int_0^c \left(\int_0^t h(u)\,du\right) h(t)\,dt \\ &= \int_0^c \left(\int_0^u h(t)\,dt\right) h(u)\,du \\ &= \int_0^c du\left(h(u)\int_u^c h(t)\,dt\right) \\ &= \tfrac{1}{2}\int_0^c du\left(h(u)\int_0^c h(t)\,dt\right) = \tfrac{1}{2}\left(\int_0^c h(t)\,dt\right)^2 = \tfrac{1}{2}[g(c)]^2 \\ &= \int_0^k f(x)\,dx.\end{aligned}$$

Here it has been tacitly assumed that change of order of integration is allowable. That it is so follows, for example, from the theorem that change of order of integration is always allowable when the integrand is positive, or when it is negative at most at points forming a plane set whose section by every ordinate and every abscissa has zero linear content. A little consideration will shew, however, that the change of order is, in our case, allowable, without recourse to this theorem.

This proves the theorem when $m = 1$. A similar proof will serve when $m = 2$. Denoting the right-hand side of the equality to be proved by X, we have

$$\begin{aligned}X &= \int_0^c [g(t)]^2\,h(t)\,dt = \int_0^c dt\,h(t)\int_0^t du\,h(u)\int_0^t dv\,h(v) \\ &= \int_0^c du\,h(u)\int_u^c dt\,h(t)\int_0^t dv\,h(v) \\ &= \int_0^c du\,h(u)\int_0^u dv\,h(v)\int_u^c dt\,h(t) + \int_0^c du\,h(u)\int_u^c dv\,h(v)\int_v^c dt\,h(t).\end{aligned}$$

Therefore adding the last two lines, after having, in the former of the two lines interchanged t and v, we have an expression for $2X$ the same as that last obtained for X, except that the last integration on the right has the limits 0 and c instead of the limits v and c.

Add to this expression the first expression obtained above for X as a

three-fold repeated integral, after having written u for t, v for u, and t for v. Then we get the same expression for $3X$ as we had got for $2X$, except that the last integration in the first term now has 0 to c as limits, instead of u to c.

In the expression last obtained the two integrals obviously add up, and we get for $3X$ a three-fold integral in which all three sets of limits of integration are 0 to c. In other words,

$$3X = [g(c)]^3.$$

This proves the theorem in the case when $n = 2$.

It is clear that the argument is generalisable.

[*Note added October*, 1910.—My object in the present paper has been, in continuation of former work of mine, to bring out more especially the use of the method of monotone sequences in the theory of integration. It is frequently of great use also in practice. Even in the present paper, however, we have had to have recourse to the theorem of Weierstrass. Lebesgue has shewn in his recent paper, quoted incidentally above, that, in the applications, sequences which are not monotone may be employed in an extraordinarily powerful as well as elegant manner. Readers interested in this subject cannot do better than study Lebesgue's admirable and strikingly original memoir.]

ON THE 3–3 BIRATIONAL TRANSFORMATION IN THREE DIMENSIONS*

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5. Specimen of the analysis.
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1.

In a paper read before the London Mathematical Society in 1869, Cayley† was the first to call attention to birational transformations in three dimensions, as an immediate extension of Cremona's two-dimensional work. In particular he investigated the 3–3 transformation, which is most compactly expressed by three bilinear relations between the two sets of homogeneous coordinates in the two spaces. He shewed that planes in one space correspond to cubic surfaces in the other, and since three planes meet in only one point, three of these cubic surfaces meet in only one variable point, and their remaining intersection is a fixed sextic curve with seven apparent double points. Cayley gave the formulæ of transformation for two cases: (i) when the sextic does not degenerate, and (ii) when it becomes four non-intersecting straight lines and their two transversals. Soon afterwards Cremona‡ and Noether§ both gave many examples proving that the sextic may degenerate in different ways in the two spaces.

The method of inversion, which is a special form of this transformation, had long been known; but since the time of Cremona the general theory

* For the suggestion of this subject I am indebted to Mr. Berry.

† Cayley, *Proc. London Math. Soc.*, Vol. III, p. 127, 1869.

‡ Cremona, *Rend. R. Ist. Lomb.*, T. IV, p. 269, 1871; *Math. Ann.*, Bd. IV, p. 213, 1871; *Ann. di Mat.*, T. V, p. 131, 1872; and other papers.

§ Noether, *Math. Ann.*, Bd. III, p. 547, 1871.

has been greatly developed, and birational transformations are constantly employed in connection with such problems as singularities of surfaces.* In particular, the 3–3 transformation, and some of its special cases, have been very useful, and it seems worth while to attempt a systematic treatment of all its varieties. Many of these have been discussed,† but, as far as I know, with no attempt at completeness.

The following paper deals with the cases where the first principal sextic curve consists of six straight lines with eight simple intersections, or their equivalent in multiple intersections, taking account of all the possible arrangements of these intersections on the lines.

The different types which arise are as follows :—

		No. of Cases.
A	8 simple intersections	5
B	1 triple and 6 simple intersections	7
B′	1 special triple and 5 simple intersections...	3
C	2 triple and 4 simple intersections	4
C′	1 special triple, 1 ordinary triple, and 3 simple intersections	2
C″	2 special triple and 2 simple intersections...	1
D	3 triple and 2 simple intersections	1
E	4 triple intersections	1
F	1 quadruple and 4 simple intersections	2
G	1 quadruple, 1 triple, and 2 simple intersections	1
H	1 quintuple and 2 simple intersections	1
	Total	28

It appears that for the purpose of this paper the following must be taken as equivalent :—

An ordinary triple intersection with 2 simple intersections,
A special „ „ „ 3 „ „
A quadruple intersection „ 4 „ „
A quintuple „ „ 6 „ „

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* The most recent paper I have seen on birational transformations in general, is by Margherita Beloch, *Ann. di Mat.*, T. XVI, p. 27, 1909.

† v. Krieg, *Zeitschrift für Math. und Phys.*, Bd. XXIX, p. 38, 1884 ; Ascioni, *Giorn. di Mat.*, T. XXXI, p. 55, 1893 ; Döhlemann, *Sitzungsber. d. K. Bay. Ak. der Wiss. zu München*, Bd. XXIV, p. 41, 1894 (among many others).

2.

If we start with the three bilinear relations between the two sets of homogeneous coordinates $xyzw$, $XYZW$ of corresponding points in the two spaces, they can be put into either of the forms

$$\left.\begin{aligned} Xp_1+Yq_1+Zr_1+Ws_1 &= 0 \\ Xp_2+Yq_2+Zr_2+Ws_2 &= 0 \\ Xp_3+Yq_3+Zr_3+Ws_3 &= 0 \end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned} xP_1+yQ_1+zR_1+wS_1 &= 0 \\ xP_2+yQ_2+zR_2+wS_2 &= 0 \\ xP_3+yQ_3+zR_3+wS_3 &= 0 \end{aligned}\right\},$$

where $p_1 \dots s_3$ are linear functions of $xyzw$, and $P_1 \dots S_3$ of $XYZW$. Solving these, $XYZW$ are proportional to the determinants of the matrix

$$\left\| \begin{matrix} p_1 & q_1 & r_1 & s_1 \\ p_2 & q_2 & r_2 & s_2 \\ p_3 & q_3 & r_3 & s_3 \end{matrix} \right\|,$$

and $xyzw$ to those of

$$\left\| \begin{matrix} P_1 & Q_1 & R_1 & S_1 \\ P_2 & Q_2 & R_2 & S_2 \\ P_3 & Q_3 & R_3 & S_3 \end{matrix} \right\|,$$

and the first sextic curve arises as the common intersection of the four cubic surfaces in the first space $0 = X = Y = Z = W$.

In what follows this process is reversed. First, I enumerated the particular forms of degenerate sextics to be considered. For each of these I wrote down the most convenient equations, and obtained by inspection four linearly independent cubic surfaces $0 = X = Y = Z = W$ containing them; in every case degenerate surfaces are available. (It is possible to express the conditions that the general cubic surface contains the sextic; these conditions can be used to reduce its twenty coefficients to linear functions of four; these four then are multiplying $XYZW$.) Then I find by inspection the three bilinear relations between $XYZW$. (Again it is possible to write down the general form $Xp+Yq+Zr+Ws$, containing sixteen arbitrary constants, and express that this is identically 0; this must leave three arbitrary constants, each multiplying the left-hand side of one of the bilinear equations sought.) Having obtained these equations I solved them for $x : y : z : w$; these are proportional to four cubic functions of $XYZW$, which equated to 0 give four cubic surfaces in the second space whose common intersection is the second sextic. One of the chief interests of the investigation lies in the extraordinary variety of forms assumed by this curve. The only constant feature, besides the essential one of having seven apparent

double points, is that in all the cases here investigated, *i.e.*, where the first sextic is six distinct straight lines, the second sextic consists partly of three straight lines. In twelve of the twenty-eight cases considered it is of the same kind as the first sextic; in two others it is a different set of six straight lines, in ten it is a conic and four straight lines and in the remaining four cases, a cubic and three straight lines.

I have given the work at length in a typical case; for the rest I have stated the bilinear relations in their simplest form and shewn each sextic in a diagram. A closed oval represents a conic and an unclosed curve a cubic. The tetrahedron of reference is shewn in each figure, the vertices being marked 1, 2, 3, 4, and where the directions of three lines that meet in a point are parallel to a plane, this is also marked by a small arc meeting the three. The intersections shewn are the actual intersections; any which would be introduced by producing lines would be apparent intersections.

3.

The distribution of the eight intersections on the six straight lines is restricted by the fact that the system lies on a cubic surface. Thus:

(i) Not more than three can lie in a plane; for if so the four linearly independent cubic functions, which are proportional to the four coordinates in the other space, equated to 0 give degenerate surfaces consisting of this plane and a quadric through the other two lines; the functions have a common linear factor which can be removed, and the transformation is not cubic but quadratic. In particular, there cannot be five intersections between four lines.

(ii) If three of the straight lines lie in a plane, all the other three must meet one or other of them; for, as above, the cubic surfaces cannot meet the plane in any other point.

(iii) Again, the six straight lines cannot all lie in two planes or on a quadric surface; for if so the quadric and an arbitrary plane form a cubic surface with four arbitrary constants containing the system, and is therefore the most general cubic that does so; the four independent cubic functions have a common quadratic factor and the transformation is linear. (The case in which a cubic surface, though containing six generators of a quadric, need not contain the quadric, does not arise, for they are three generators of one system and three of the other, involving nine intersections.)

4.

The general cubic surface will be taken in the form

$$Ax^3+Bx^2y+Cxy^2+Dy^3+z(Ex^2+Fxy+Gy^2)+w(Hx^2+Jxy+Ky^2)$$
$$+z^2(Lx+My)+zw(Nx+Py)+w^2(Qx+Ry)$$
$$+Sz^3+Tz^2w+Uzw^2+Vw^3=0.$$

It meets any straight line in three points, and if it meet it in a fourth it contains the whole line; therefore to contain a straight line imposes four conditions. If the line (a) is $0=z=w$, the four conditions are

$$A=B=C=D=0.$$

To contain two straight lines imposes eight conditions, unless the lines intersect, when one condition is the same for both, viz., that their common point lies on the surface. If the second line (b) is $0=y=w$, the three further conditions are $E=L=S=0$. Similarly for every intersection a condition is lost. Six non-intersecting straight lines would impose twenty-four conditions; but since there are fifteen real or apparent double points, if the set of six straight lines is a particular case of Cayley's sextic curve with seven apparent double points, there are eight intersections, reducing the number of conditions to sixteen, and leaving four independent coefficients. There are therefore four linearly independent cubic surfaces through the six straight lines; these, or four linear functions of them, are proportional to the second set of coordinates.

A. *Eight simple intersections.*

Of the six lines *abcdef*, let *a* be the one that meets most. It cannot meet them all, for if so there would be three intersections between *bcdef*; one (b) would meet two others (cd) and *abcd* would lie in a plane.

(1) If *a* meets *bcde* (and not *f*), *f* may meet *bcde*, or

(2) *f* may meet *bcd* and *e* meet *b*, or

(3) *f* may meet *bc*; then *bc* cannot meet (for if so *abcf* lie in a plane), nor can *d* (or *e*) meet both of them (for if so *abcd* lie in a plane); therefore *d* meets *b*, and *e* meets *c*.
And *f* cannot meet only *b*, for if so there are three intersections between *bcde*, one (b) must meet two others (cd) and *abcd* lie in a plane.

If *a* meets *bcd* (and not *e* or *f*) (*i.e.*, considering the case where no line meets more than three), then if *bcd* do not meet each other, either

(4) *e* and *f* do not meet, one (e) meets all of *bcd* and *f* meets *bc*, or

(5) *ef* meet, and each meets two of *bcd*; say, *e* meets *bc*; then *f* cannot meet the same two (for if so *bcef* lie in a plane) but meets *bd*.

And if *bc* meet, they lie in a plane with *a*; *def* all meet this plane on one of the lines; but *a* meets *d*, and *bc* are only to meet one more each; therefore *b* meets *e*, and *c* meets *f*, and there are two intersections between *def*. But this configuration is the same as in (5) where *bef* form the triangle.

B. *One triple and six simple intersections.*

If *abc* meet in a point, they may or may not lie in a plane.

If they do not, let their equations be $0 = z = w$, $0 = y = w$, $0 = y = z$. To contain these imposes ten conditions on the cubic

$$A = B = C = D = E = L = S = H = Q = V = 0,$$

so that this singularity counts as two simple intersections and one apparent double point, and it can be obtained from A by letting two simple intersections come together. It is a conical point on all the cubic surfaces.

No one of *def* can meet all three arms of the triple point without passing through the vertex and forming a quadruple point (F below), and no two of *def* can meet the same two arms without both lying in their plane.

(1) If *def* do not meet each other, each meets a different pair of arms.

(2) If *e* meets *f* (and neither meets *d*), since one of them (*e*) must meet two arms (*ac*), *f* can only meet *b* (for if it met *a*, *acef* would lie in a plane), and then *d* must meet *b* and *a* (or *c*).

If *f* meets *de* (but *de* do not meet), there are four intersections between *def* and the arms, either two on two of *def* or two on one and one on each of the others.

(3) If *f* meets *bc*, then *d* and *e* must each meet *a*, for they cannot meet *b* or *c*.

(4) If *f* meets *c* only, one of *d* or *e* must meet *a* and *b*, and the other *a* (or *b*).

(5) If *f* meets no arm, *d* and *e* meet different pairs of arms.

If *def* form a triangle, all three arms meet one or other of them, either:

(6) each arm meets a different side, or

(7) two arms meet one side and the third meets another side.

B′. *One special triple and five simple intersections.*

If the three arms of the triple point lie in a plane, let their equations be

$$0 = z = w, \qquad 0 = y = w, \qquad 0 = w,\ y = z.$$

There are only nine conditions

$$A = B = C = D = E = L = S = F = G + M = 0.$$

The reason for this is that all the cubic surfaces meet the plane ab in a, b and a third straight line. If two other points in the plane lie on all the surfaces, the straight line c joining them is part of the common intersection ; so c only gives two more conditions and must be considered as meeting both a and b.

Such a triple intersection will be called *special;* it is not a conical point on the surfaces. It is equivalent to three simple intersections, and can be obtained from A by making a triangle diminish to a point, the sides remaining in a plane.

Now def must all meet the plane abc on one of the arms, and none of them can meet two arms without making a fourth line in the plane ; therefore they have two intersections among themselves, say df, ef. Then a cannot meet def, for if so the four lie in a plane ; therefore either

(1) a meets de, and f meets b, or

(2) a meets df, and e meets b, or

(3) a meets d, and e meets b, and f meets c.

C. *Two triple and four simple intersections.*

If neither triple point is special, each is a conical point on all the cubic surfaces, the line joining them lies on all the cubics and the triple points have a common arm ; let them be abc, ade.

Now b (or c) cannot meet both d and e, for if so $abde$ would lie in a plane ; therefore there are at most two intersections between the arms, bd, ce, and at least two on f. Also f cannot meet a, for if so it must also meet another (b) ; then neither f nor b can meet any other, and c would have to meet both d and e, which is impossible ; therefore either

(1) f meets $bcde$, or

(2) f meets bcd, and e meets b (or c), or

(3) f meets bc, one of these (b) meets d, and c meets e, or

(4) f meets bd, b meets e, and c meets d.

These cases can be obtained from B.

C′. *One special triple, one ordinary triple, and three simple intersections.*

The special triple point is not a conical point on the surfaces, and the two need not have a common arm. If they do, let them be (*abc*), *ade*; then neither *d* nor *e* can meet the plane *abc* again, all the three intersections must lie on *f*, viz., one where it meets the plane *abc*, and two with *d*, *e*. But then the whole set lies in the two planes *abc*, *def*, and the transformation degenerates; therefore the triple points have not a common arm. Let them be (*abc*), *def*; then *def* each meet one or other of *abc*. Either

(1) each meets a different one, or

(2) two meet *a* and one meets *b*.
For *def* cannot all meet *a* without the four lying in a plane.

These cases can be obtained from B or B′.

C″. *Two special triple and two simple intersections.*

The two must have a common arm, for if not the six lines would lie in the two planes. Let them be (*abc*), (*ade*). None of these can meet again, for if so all five would lie in a plane; therefore there are two intersections on *f*, where it meets the two planes. *f* cannot meet *a*, for if so it could meet neither plane again; therefore it meets *b* (or *c*) and *d* (or *e*).

This case can be obtained from B′.

D. *Three triple and two simple intersections.*

With three triple points and only six lines, each pair must have a common arm, and three of the lines are the sides of the triangle of triple points. None of them can be special, for if so there would be a fourth line in the plane of the triangle. There must be two intersections between the remaining three arms.

This case can be obtained from C.

E. *Four triple intersections.*

As in the last case, no one can be special, and the six lines are the six edges of the tetrahedron of triple points.

This case can be obtained from D.

If four straight lines meet in a point, and no three lie in a plane, let their equations be

$$0 = z = w, \qquad 0 = y = w, \qquad 0 = y = z, \qquad y = z = w,$$

giving the twelve conditions

$$A = B = C = D = E = L = S = H = Q = V = F+J+N$$
$$= G+K+M+P+R+T+U = 0,$$

so that this singularity is equivalent to four simple intersections. It cannot, in general, be obtained from B by making a fourth line pass through the triple point, as that would only absorb three intersections, except in the two cases where (i) the triple point was special, (ii) the fourth line met two of the arms; each of these leads to a special quadruple point with three arms in a plane. Now, if three arms lie in a plane, let the equations of the last be replaced by $y = z$, $0 = w$; then the last two conditions are replaced by two others, viz., $F = G+M = 0$, so in this case also the quadruple point is equivalent to four simple intersections, we have a particular case and not a fresh type. The analytical view of this is as follows:—

Each straight line through the vertex (1, 0, 0, 0) imposes one condition on the coefficients of each power of x. Now, there are: one term in x^3, three in x^2, six in x^1, ten in x^0. The first straight line imposes four conditions and abolishes the term in x^3. The second and third impose three conditions each, and abolish the terms in x^2, unless the three lines lie in a plane, when the three conditions on the coefficients of terms in x^2 reduce to two (and the factor multiplying x^2 gives the equation of the plane). With four arms, which cannot all lie in a plane, taking first three which do not, they impose ten conditions and abolish the terms in x^3, x^2. The fourth, whether it lie in a plane with two of the others or not, imposes two conditions, on the coefficients of terms in x^1, x^0 respectively. A fifth and sixth also impose two conditions each, and abolish the terms in x^1 (in which case x has disappeared entirely, the cubic surfaces are all cones, and cannot have a single variable point of intersection: the transformation breaks down); except in the case where the sixth straight line lies on the quadric cone through the first five, given by the factor multiplying x. But in this case the six lines lie on a quadric and do not give rise to a cubic transformation. Therefore a sextuple point need not be considered.

F. *One quadruple and four simple intersections.*

Let the quadruple point be *abcd*. Then neither *e* nor *f* can meet three arms.

(1) If they do not meet each other, *ef* each meet two arms; not the same pair, for if so the four lie in a plane; and not distinct pairs, *ab*, *cd*, for if so the whole lies in the two planes *abe*, *cdf*; therefore one (*e*) meets *ab* and *f* meets *ac*.

(2) If *ef* meet, one (*e*) meets two arms and lies in their plane, and *f* meets a third arm.

These cases cannot be obtained from former ones.

G. *One quadruple, one triple, and two simple intersections.*

Let these be *abcd*, *aef*, for there must be a common arm. Then *e*, *f* each meet a different one of *bcd*. The triple point cannot be special, for if so one intersection would make four lines lie in a plane.

This case can be obtained from F.

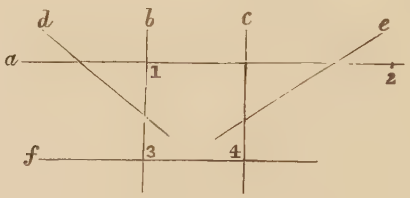
H. *One quintuple and two simple intersections.*

If the quintuple point is *abcde*, then *f* meets two of the arms.

This case cannot be obtained from any former one.

5.

I give a specimen of the work in detail.



With this set of intersections, taking the tetrahedron of reference as shewn, the equations of the six lines can be

$$(a)\ z = w = 0, \qquad (b)\ y = w = 0, \qquad (d)\ w = lx + my + nz = 0,$$

$$(f)\ x = y = 0, \qquad (c)\ z = \alpha x + \beta y = 0, \qquad (e)\ z = px + qy + sw = 0.$$

By inspection four sets of three planes are found to contain these, and we write

$$\frac{X}{yzw} = \frac{Y}{xzw} = \frac{Z}{w(\alpha x + \beta y)(px + qy + sw)} = \frac{W}{yz(lx + my + nz)}.$$

These satisfy the three bilinear relations

$$Xx = Yy, \qquad Zz = (\beta X + \alpha Y)(px + qy + sw), \qquad Ww = X(lx + my + nz).$$

Solving these, we find

$$\frac{x}{Y[ZW - nsX(\beta X + \alpha Y)]} = \frac{y}{X[ZW - nsX(\beta X + \alpha Y)]} = \frac{z}{(\beta X + \alpha Y)[W(qX + pY) + sX(mX + lY)]}$$

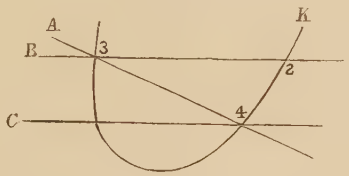
$$= \frac{w}{X[Z(mX + lY) + n(\beta X + \alpha Y)(qX + pY)]}.$$

The denominators equated to 0 give four cubic surfaces whose common intersection is the second sextic. This is found to consist of three straight lines A, B, C and a cubic K, the equations being

$$(A)\ X = Y = 0, \qquad (B)\ X = W = 0, \qquad (C)\ Z = \beta X + \alpha Y = 0,$$

and B, K are the complete intersection of the two quadrics

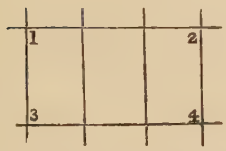
$$W(qX + pY) + sX(mX + lY) = 0, \qquad ZW = nsX(\beta X + \alpha Y).$$



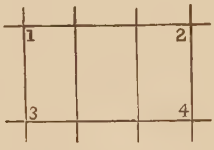
In this, putting $\alpha = 0$, abc, BCK concur (B1); $p = 0$, abe concur and K degenerates (B2); $\alpha = s = 0$, abc, cef concur (C3), &c.; $n = m = q = \alpha = s = 0$, abe, cef, ade, bdf concur (E). In the last case the sextic in each space consists of the six edges of a tetrahedron, and the transformation is the well known one of inversion.

6.

A 1.

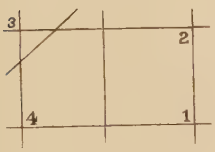


$$X(z + w) = Z(ax + \beta y)$$
$$W(x + y) = Y(\gamma z + \delta w)$$
$$(X + \gamma Y)z = (aZ + W)x$$

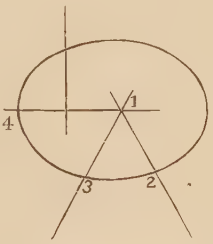


Cayley, *Proc. London Math. Soc.*, Vol. III, p. 174; Noether, *Math. Ann.*, Bd. III, p. 571; Cremona, *ib.* Bd. IV, p. 221; v. Krieg, *Zeitschr. für Math.*, Bd. XXIX, p. 60; Ascioni, *Giorn. di Mat.*, T. XXXI, p. 79.

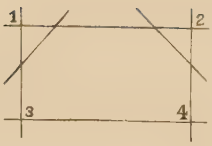
A 2.



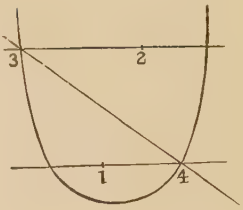
$$Zz = Ww$$
$$Y(y + z) = W(w + \delta x)$$
$$[\delta(rW + sZ) - X]x = (Y - Z)(qy + rz + sw)$$



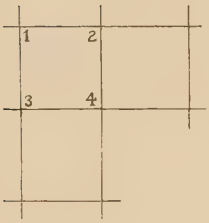
A 3.



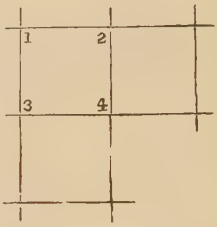
$$Xx = Yy$$
$$Zz = Y(px + qy + sw)$$
$$Ww = X(lx + my + nz)$$



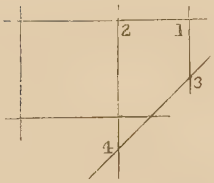
A 4.



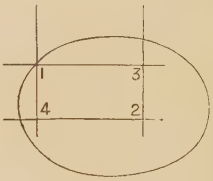
$$(\alpha X - \gamma Z)(x + z) + (\beta Y - \delta W)(y + w) = 0$$
$$(X - Y)(\alpha x + \beta y) + (Z - W)(\gamma z + \delta w) = 0$$
$$X(\alpha z - \beta y) + \beta Y(y + w) - Z(\gamma z + \delta w) = 0$$



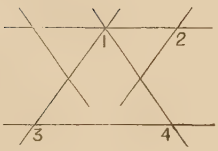
A 5.



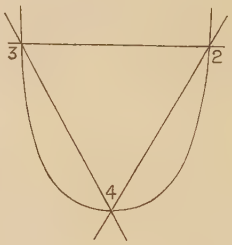
$$Yy = Ww$$
$$Z(z + w) = Y(x + y + z + \delta w)$$
$$Xx + Zy = W(x + y + z + \delta w)$$



B 1.

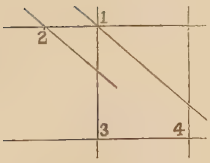


$$Xx = Yy$$
$$Zz = X(px + qy + sw)$$
$$Ww = X(lx + my + nz)$$

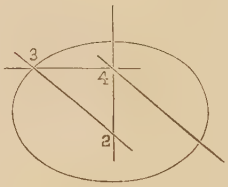


Cremona, *Rend. R. Ist. Lomb.*, T. IV, p. 319.

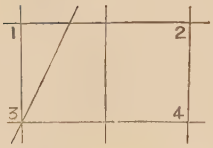
B 2.



$$Xx = Yy$$
$$Zz = (\beta X + \alpha Y)(y + w)$$
$$Ww = X(x + z)$$

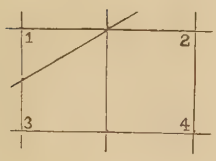


B 3.

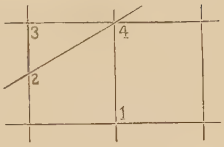


$$Xx = Zz$$
$$Y(x + y) = Z(z + \delta w)$$
$$(\delta pZ - W)w = (Y - X)(px + qy)$$

B 4.



$$Xx = Zz$$
$$Y(x+y) = Z(z+\delta w)$$
$$[\delta(X+Z)+W]\,w = (Y-X)(x+y+z)$$



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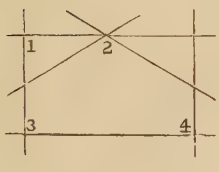
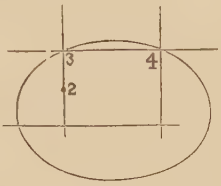


$$Xx = Yy$$
$$Zz = (X+Y)(x+w)$$
$$Ww = X(x+z)$$



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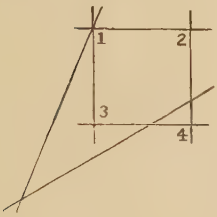
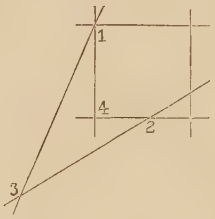
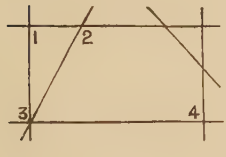


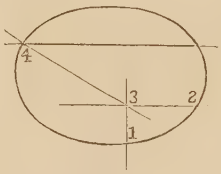
$$Yy = Ww$$
$$Z(z+w) = Y(y+z+\delta w)$$
$$Xx + Zy = W(y+z+\delta w)$$



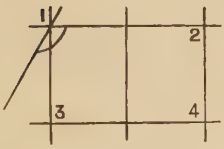
B 7.



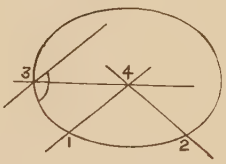
$$Xx = Yy = Ww$$
$$Zz = (X+Y)(ax+y+w)$$



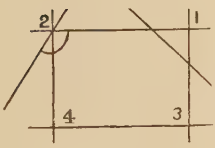
B' 1.



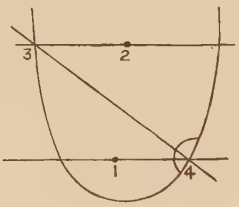
$$Xx = Zz$$
$$Y(x+y) = Z(z+w)$$
$$(X+W)\,w = (Y-X)(y+z)$$



B' 2.



$$Xx = Yy$$
$$Zz = Y(x+w)$$
$$Ww = X(x+y+z)$$



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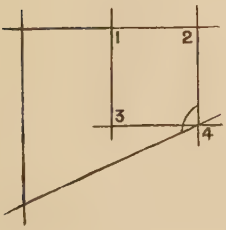
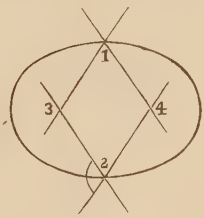
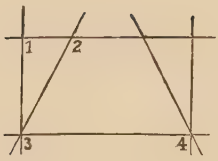


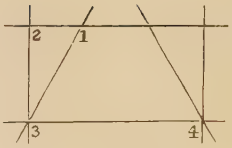
$$Yy = Ww$$
$$Z(z+w) = Y(x+y+z)$$
$$Xx + Zy = W(x+y+z)$$



C 1.

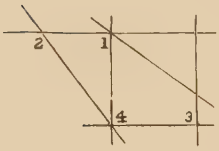


$$Xx = Yy = Ww$$
$$Zz = (X+Y)(px+qy)$$

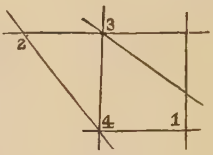


Ascioni, *Giorn. di Mat.*, T. XXXI. p. 69.

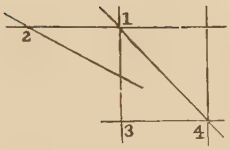
C 2.



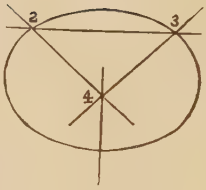
$$Xx = Yy = Zz$$
$$Ww = (X-Y)(y+z)$$



C 3.



$$Xx = Yy$$
$$Zz = X(x+y)$$
$$Ww = X(x+z)$$



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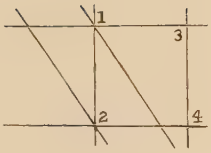
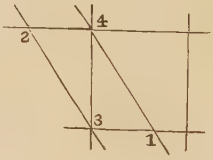
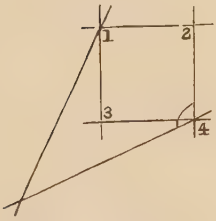


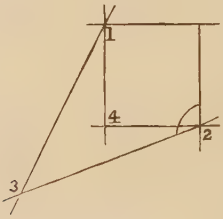
$$Xx = Yy$$
$$Zz = Y(y+w)$$
$$Ww = X(x+z)$$



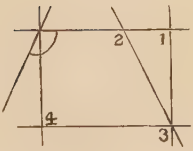
C' 1.



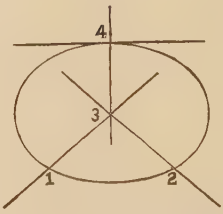
$$Yy = Ww$$
$$Z(z+w) = Y(y+z)$$
$$Xx + Zy = W(y+z)$$



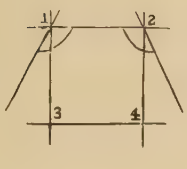
C' 2.



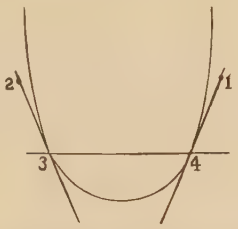
$$Xx = Yy = Ww$$
$$Zz = (X+Y)(x+y+w)$$



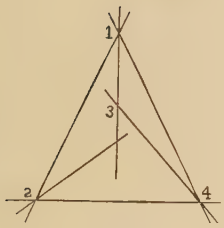
(A straight line touches the conic.)

C″. 

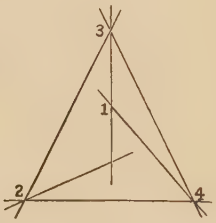
$$Xx = Yy$$
$$Zz = Y(x+w)$$
$$Ww = X(y+z)$$

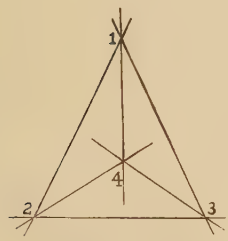


(Two straight lines touch the cubic.)

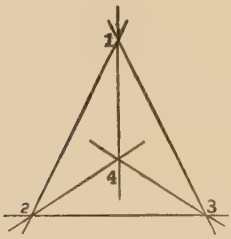
D. 

$$Xx = Yy = Zz$$
$$Ww = X(x+z)$$

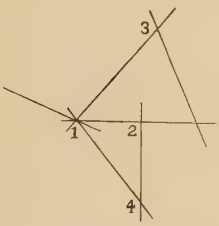


E. 

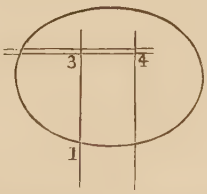
$$Xx = Yy = Zz = Ww$$



Inversion : passim.

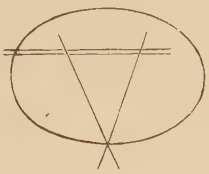
F 1. 

$$Zz = (Y - \beta X)\,x$$
$$Ww = Y(x+y)$$
$$X(y+\beta w) = Y(z+w)$$



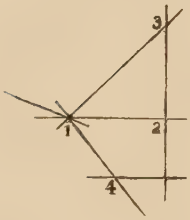
(Two straight lines coincide.)

$\beta = 0$

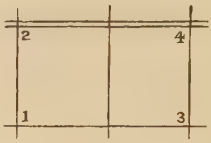


If in the first figure the three lines lie in a plane, in the second two lines meet on the conic.

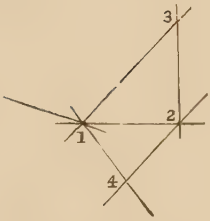
F 2.



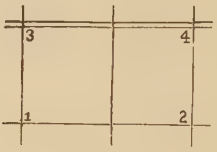
$$Xx = Y(\gamma y + z)$$
$$X(z + w) = Z(y + \beta z)$$
$$Ww = \beta Zx - Y(z + w)$$



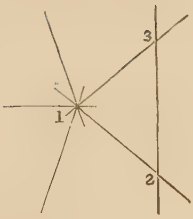
G.



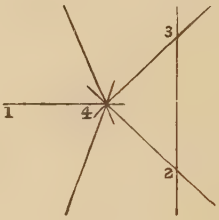
$$Xx = Zz$$
$$Y(y + \beta z) = X(z + w)$$
$$Ww = Z(z \cdot w) - \beta Yx$$



H.



$$Xx = Ww$$
$$Yy = Zz$$
$$X[(1-\gamma)y + (1-\beta)z + (1-\beta\gamma)w]$$
$$= (1-\beta\gamma)Zz + [\beta(1-\gamma)Y + \gamma(1-\beta)Z]w$$



$\gamma = 0$

If in the first figure three lines lie in a plane, in the second two lines coincide.

$\beta = \gamma = 0$

THE SCATTERING OF LIGHT BY A LARGE CONDUCTING SPHERE

By J. W. NICHOLSON.

[Received February 24th, 1910.—Read March 10th, 1910.]

WHEN plane waves fall on a sphere of perfect conductivity, it is known that the character of the secondary disturbance due to the obstacle is determined by the relative magnitudes of the wave length of light and the radius of the sphere. Investigators hitherto have confined their attention, in the case of a plane wave train, to obstacles of small size in comparison with the wave length. The most exhaustive direct treatment of the problem appears to be that of Sir J. J. Thomson,* who has expressed the Cartesian components of the electric polarisation radiated from the sphere in the forms of harmonic series. These series are very cumbrous in form, and, although they readily yield a physical solution for a very small obstacle, their use has hitherto been restricted to this special case. The special problem of a very small obstacle was first solved by Lord Rayleigh,† and has been the subject of a more detailed treatment, based on Lamb's‡ general solution of the equations of wave motion, by Love.§ But for the other extreme case in which the obstacle is very large, which has many important applications, no solution has been given. In the present communication, an examination of the effect radiated by such an obstacle is given, restriction being made to points of space whose distance from the obstacle is very great in comparison with its radius, which itself bears a large ratio to the wave length of the incident train. The series given by Sir J. J. Thomson are taken as the starting-point of the investigation. The results are very different from those of a corresponding small obstacle. In the latter case, the mean energy scattered by the sphere is closely dependent on the wave length, being inversely proportional to its fourth power. But when the radius is a large multiple of the wave length, this dependence may be expected to vanish. This conclusion is confirmed by the results of the paper.

* *Recent Researches in Electricity and Magnetism.*

† *Phil. Mag.*, August, 1881, Scientific Papers, I, pp. 518 *et seq.*

‡ *Proc. London Math. Soc.*, Vols. XIII, XV.

§ *Proc. London Math. Soc.*, Vol. XXX.

A system of plane waves, travelling along the direction x and incident on the positive side, may be represented by an electric polarisation of components, along (x, y, z) axes respectively,

$$f_0 = 0, \qquad g_0 = 0, \qquad h_0 = e^{\iota kx}, \tag{1}$$

with a magnetic force

$$\alpha_0 = 0, \qquad \beta_0 = -4\pi V e^{\iota kx}, \qquad \gamma_0 = 0, \tag{2}$$

ignoring a time factor.

The wave length of the vibration is $2\pi/k$, and V is the velocity of propagation of electromagnetic waves in the external medium. Writing, in terms of spherical polar coordinates with origin at the centre of the sphere,

$$x = r\cos\theta, \qquad y = r\sin\theta\cos\phi, \qquad z = r\sin\theta\sin\phi, \tag{3}$$

then, corresponding to this plane wave system, Sir J. J. Thomson's series for the polarisation radiated outward from the sphere become, with a change of x to $-x$,

$$\begin{aligned}(f, g, h) &= \frac{1}{\iota k}\sum_1^\infty \frac{\iota^n}{n.n+1}\left\{f_{n-1}(kr)\,\frac{\partial}{\partial x, y, z}\left(r^n\sin\theta\sin\phi\,\frac{dP_n}{d\mu}\right)\right.\\ &\qquad \left. -nk^2r^{2n+3}f_{n+1}(kr)\,\frac{\partial}{\partial x, y, z}\left(\sin\theta\sin\phi\,r^{-n-1}\frac{dP_n}{d\mu}\right)\right\}\frac{\dfrac{\partial}{\partial a}\,aS_n(ka)}{\dfrac{\partial}{\partial a}\,a^{n+1}f_n(ka)}\\ &\quad -\sum_1^\infty \frac{2n+1.\iota^n}{n.n+1.a^n}\,\frac{S_n(ka)}{f_n(ka)}\,f_n(kr)\left\{y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y},\quad z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z},\quad x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right\}\\ &\qquad \times r^n\sin\theta\cos\phi\,\frac{dP_n}{d\mu},\end{aligned}$$

where

$$\left.\begin{aligned} e^{\iota kr\mu} &= \sum_0^\infty 2n+1.\iota^n S_n(kr)P_n(\mu),\\ f_n(kr) &= \left(\frac{1}{kr}\frac{\partial}{\partial kr}\right)^n\frac{e^{-\iota kr}}{kr},\end{aligned}\right\} \tag{4}$$

μ is $\cos\theta$, and $P_n(\mu)$ the ordinary zonal harmonic of order n. The radius of the sphere is a.

It will be found more convenient to use the ordinary notation for Bessel functions in place of $S_n(kr)$ and $f_n(kr)$. If $J_m(\rho)$ denote the Bessel function of the first kind of argument ρ, where $m = n+\frac{1}{2}$, then

$$J_m(\rho) = \frac{\rho^m}{2^m.\Gamma(m+1)}\left\{1-\frac{\rho^2}{2^2.1!.m+1}+\frac{\rho^4}{2^4.2!.m+1.m+2}-\ldots\right\}, \tag{5}$$

and $J_{-m}(\rho)$ is the same function with the sign of m changed. As a function suitable for the expression of a divergent wave, we introduce*

$$K_m(\rho) = \frac{\pi}{2 \sin m\pi} \left\{ e^{-\frac{1}{2}\iota m\pi} J_{-m}(\rho) - e^{\frac{1}{2}\iota m\pi} J_m(\rho) \right\}. \tag{6}$$

It may be shewn at once that

$$S_n(kr) = \left(\frac{\pi}{2kr}\right)^{\frac{1}{2}} J_m(kr), \tag{7}$$

where m here, and throughout the paper, denotes $n+\frac{1}{2}$.

Similarly, both being functions appropriate to a diverging wave,

$$a^{n+1} f_n(ka) = \iota k^{-n-1} \left(\frac{2ka}{\pi}\right)^{\frac{1}{2}} K_m(ka) e^{-\frac{1}{2}\iota m\pi}. \tag{8}$$

Writing, in the subsequent reduction,

$$ka = z, \tag{9}$$

which will cause no confusion with the Cartesian z, then

$$k^n a^n f_{n-1}(ka) + k^{n+2} a^{n+2} f_{n+1}(ka) = -\left(\frac{2z}{\pi}\right)^{\frac{1}{2}} \left\{ K_{n-\frac{1}{2}}(z) - K_{n+\frac{3}{2}}(z) \right\} e^{-\frac{1}{2}\iota m\pi}.$$

But the recurrence formula for functions of type K is found to be, when $m-\frac{1}{2}$ is an integer,

$$K_{n-\frac{1}{2}}(z) - K_{n+\frac{3}{2}}(z) = \frac{2\iota m}{z} K_{n+\frac{1}{2}}(z), \tag{10}$$

so that $k^n a^n f_{n-1}(ka) + k^{n+2} a^{n+2} f_{n+1}(ka) = -2\iota m \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} K_m(z) e^{-\frac{1}{2}\iota m\pi}$,

and, in a similar manner,

$$n+1 \,.\, r^n f_{n-1}(kr) - nk^2 r^{n+2} f_{n+1}(kr) = 2\iota m k^{-n} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\partial}{\partial kr} (kr)^{\frac{1}{2}} K_m(kr) e^{-\frac{1}{2}\iota m\pi},$$

and the formulæ may be at once transformed if desired.

In order to separate the oscillating portions of the Bessel functions, we now write, introducing two functions R_n and ϕ_n of z,

$$\left.\begin{aligned} J_m(z) &= \left(\frac{2R_n}{\pi z}\right)^{\frac{1}{2}} \sin \phi_n, \\ J_{-m}(z) &= (-)^n \left(\frac{2R_n}{\pi z}\right)^{\frac{1}{2}} \cos \phi_n, \end{aligned}\right\} \tag{11}$$

and, from the identical relation satisfied by the Bessel functions, it appears that

$$\frac{\partial \phi_n}{\partial z} = R_n^{-1}. \tag{12}$$

* Macdonald, *Proc. London Math. Soc.*, Vol. XXXII.

First approximations to R_n and ϕ_n in this special case were given by Lorenz,* and their values for various ranges of the ratio n/z have been developed in a series of papers by the writer.†

Introducing also a new function χ_n of z defined by

$$\tan \chi_n = -\tfrac{1}{2} \frac{\partial R_n}{\partial z}, \tag{13}$$

the following results may be proved, after some reduction,

$$z^{\frac{1}{2}} K_m(z) = \left(\frac{\pi R_n}{2}\right)^{\frac{1}{2}} e^{-\iota m\pi - \iota\phi_n},$$

$$z^{\frac{1}{2}} K_m(z) \Big/ \frac{d}{dz}\, z^{\frac{1}{2}} K_m(z) = \iota k^{-1} R_n \cos \chi_n e^{\iota\chi_n},$$

$$\frac{d}{dz}\, z^{\frac{1}{2}} J_m(z) = \left(\frac{2}{\pi R_n}\right)^{\frac{1}{2}} \cos(\phi_n + \chi_n) \sec \chi_n.$$

If R_{n_r}, ϕ_{n_r}, χ_{n_r} are derived in the same manner from the Bessel functions of argument kr, and if

$$\tan \chi_{n_r} = -\tfrac{1}{2} \frac{\partial}{\partial kr} R_{n_r}, \tag{14}$$

then $$\frac{\partial}{\partial kr} (kr)^{\frac{1}{2}} K_m(kr) = -\iota \left(\frac{\pi}{2R_{n_r}}\right)^{\frac{1}{2}} \sec \chi_{n_r} e^{-\iota m\pi - \iota\phi_{n_r} + \iota\chi_{n_r}},$$

and for points at a very great distance from the obstacle, it is known that

$$\left.\begin{aligned} R_{n_r} &= \frac{kr}{(k^2r^2 - m^2)^{\frac{1}{2}}} = 1, \\ \phi_{n_r} &= kr - \tfrac{1}{2}n\pi, \end{aligned}\right\} \tag{15}$$

and therefore $$\chi_{n_r} = 0.$$

Finally, for distant points, we may write

$$n+1 \,.\, r^n f_{n-1}(kr) - nk^2 r^{n+2} f_{n+1}(kr) = 2\iota m k^{-n} e^{-\iota m\pi - \iota kr},$$

$$r^n f_{n-1}(kr) + k^2 r^{n+2} f_{n+1}(kr) = -2mr^{-1} k^{-n-1} e^{-\iota m\pi - \iota kr}.$$

Moreover, on reduction,

$$\frac{\partial}{\partial a}\, a S_n(ka) \Big/ \frac{\partial}{\partial a}\, a^{n+1} f_n(ka) = k^n \cos(\phi_n + \chi_n) e^{\frac{1}{2}\iota m\pi + \iota\phi_n + \iota\chi_n},$$

$$\frac{S_n(ka)}{a^n f_n(ka)} = \iota k^n \sin \phi_n e^{-\iota m\pi + \iota\phi_n},$$

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* *Œuvres Scientifiques*, Vol. I, pp. 435 *et seq.*

† *Phil. Mag.*, Dec., 1907; Aug., 1908; July, 1909; Feb., 1910. Also *Brit. Assoc. Report*, Dublin, 1908.

and thus, after rearrangement of the harmonic series,

$$(f, g, h)$$

$$= (x, y, z) \sum_1^\infty \Big[k^{-1} \iota^{n-1} r^{n-2} \{ f_{n-1}(kr) + k^2 r^2 f_{n+1}(kr) \} \sin\theta \sin\phi \frac{dP_n}{d\mu}$$

$$+ (kn.n+1)^{-1} \iota^{n-1} \{ n+1 . r^n f_{n-1}(kr) - nk^2 r^{n+2} f_{n+1}(kr) \}$$

$$\times \frac{\partial}{\partial x, y, z} \sin\theta \sin\phi \frac{dP_n}{d\mu} \Big] \frac{\frac{\partial}{\partial a} a S_n(ka)}{\frac{\partial}{\partial a} a^{n+1} f_n(ka)}$$

$$- \sum_1^\infty \frac{\iota^n (2n+1)}{a^n n.n+1} r^n f_n(kr) \frac{S_n(ka)}{f_n(ka)} \left\{ y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}, \ \ldots, \ \ldots \right\} \sin\theta \cos\phi \frac{dP_n}{d\mu}$$

$$= - \frac{(x, y, z)}{k^2 r^3} \sum_1^\infty 2m \cos(\phi_n + \chi_n) \sin\theta \sin\phi \frac{dP_n}{d\mu} e^{\iota\phi_n + \iota\chi_n + \iota n\pi - \iota kr}$$

$$+ \iota \sum_1^\infty \frac{2m}{kn.n+1} \cos(\phi_n + \chi_n) \frac{\partial}{\partial x, y, z} \sin\theta \sin\phi \frac{dP_n}{d\mu} e^{\iota\phi_n + \iota\chi_n + \iota n\pi - \iota kr}$$

$$- \sum_1^\infty \frac{2m}{krn.n+1} \sin\phi_n \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}, \ \ldots, \ \ldots \right) \sin\theta \sin\phi \frac{dP_n}{d\mu} e^{\iota\phi_n + \iota n\pi - \iota kr}. \quad (16)$$

These three series will be denoted by S_1, S_2, S_3 respectively.

The differentiations may now be taken outside the signs of summation on the understanding that they act only on θ and ϕ, and not on r.

The Values of R_n, ϕ_n, and χ_n.—The orders of magnitude of R_n, ϕ_n, and χ_n, in so far as they are required for the present problem, have been calculated by Macdonald* from the first approximations of Lorenz,† and it is sufficient to quote the main results at this point.

The approximate formulæ are as follows:—

(1) when $z-m$ is positive, and not of order $z^{\frac{1}{3}}$,

$$R_n = \frac{z}{(z^2-m^2)^{\frac{1}{4}}}, \qquad \phi_n = (z^2-m^2)^{\frac{1}{2}} + m \sin^{-1}\frac{m}{z} - \tfrac{1}{2}n\pi; \qquad (17)$$

(2) when $m-z$ is positive and not of order $z^{\frac{1}{3}}$,

$$R_n = 2T_n \cosh 2t_n, \qquad \tan\phi_n = e^{2t_n},$$

where

$$2T_n = \frac{z}{(m^2-z^2)}, \quad t_n = -\tfrac{1}{2}\log 2 + (m^2-z^2)^{\frac{1}{2}} + m \log\{m - (m^2-z^2)^{\frac{1}{2}}\} - m\log z; \qquad (18)$$

* *Proc. Royal Soc.*, 1904. † *Loc. cit.*

(3) when $m-z$ or $z-m$ is of order not greater than $z^{\frac{1}{3}}$,

$$\left.\begin{aligned} R_n &= 3^{-\frac{5}{6}}\pi^{-\frac{1}{2}}z^{\frac{1}{3}}\left\{\pi\left(-\tfrac{5}{6}\right)+(m-z)\,\pi\left(-\tfrac{1}{2}\right)\left(\frac{24}{m}\right)^{\frac{1}{3}}+\ldots\right\}, \\ \phi_n &= \tfrac{1}{6}\pi+(R_n)^{-1}_{m=z}(z-m), \end{aligned}\right\} \tag{19}$$

so that ϕ_n ceases to be large with z in this region.

From these formulæ it is possible to prove that χ_n is an angle which is extremely small (of order z^{-1}) in the first region, becoming $\frac{1}{6}\pi$ at the critical point $m=z$, and attaining the value $\frac{1}{2}\pi$ in the second region. Its variation between these values is continuous, and it never becomes large.

Moreover, $\partial\phi_n/\partial m$ is always negative. In the second region it is very small, and near $m=z$, its order is that of $m^{-\frac{1}{3}}$, so that only in the first region is it comparable with unity. $\partial\chi_n/\partial n$ is never of higher order than $m^{-\frac{1}{3}}$.

In the succeeding analysis we shall write $m/z=t$ so that, in the only case in which $\partial\phi_n/\partial t$ is of order z,

$$R_n=(1-t^2)^{-\frac{1}{2}}, \qquad \phi_n=z\left\{(1-t^2)^{\frac{1}{2}}+t\sin^{-1}t-\tfrac{1}{2}\pi t\right\}+\tfrac{1}{4}\pi, \tag{20}$$

and χ_n may be neglected, together with $\partial\chi_n/\partial t$.

Summation of the Series S_1.—When θ is not nearly equal to zero or π, thus excluding the diameter of the sphere parallel to the incident rays, the derivate of the zonal harmonic may be replaced by its asymptotic expansion. For it will subsequently appear that the most important harmonics in the summation are those whose order n is comparable with z.

Thus
$$\frac{dP_n}{d\mu}=\left(\frac{2m}{\pi\sin^3\theta}\right)^{\frac{1}{2}}\sin\left(m\theta-\tfrac{1}{4}\pi\right), \tag{21}$$

with this substitution, it appears that the oscillating factors of the three series S_1, S_2, S_3 are derived from

$$(\cos\phi_n,\ \sin\phi_n)\sin\left(m\theta-\tfrac{1}{4}\pi\right)e^{\iota\phi_n+\iota m\pi}.$$

Each series therefore involves four exponential functions of arguments $\iota z(v_1, v_2, v_3, v_4)$ respectively, where

$$\left.\begin{aligned} z(v_1, v_2) &= 2\phi_n+m\pi\pm m\theta, \\ z(v_3, v_4) &= m\pi\pm m\theta, \end{aligned}\right\} \tag{22}$$

and the four exponents are all large when θ is not nearly zero or π. χ_n, as was shewn above, does not lead to rapid oscillation of the same type in passing from term to term of the harmonic series.

Now it may be shewn, as in a paper by the writer,* that the sum of a series

$$S = \sum_{n_1}^{n_2} u \left(\frac{n}{z}\right) e^{\iota z v(n/z)}, \tag{23}$$

u and v being functions of $n/z \equiv t$, where n takes values whose differences are integers, and where differentiation with respect to t does not raise the order of u or v in the large quantity z, is, rejecting terms of relative order z^{-1}, at least,

$$S = z \int_{\epsilon_1}^{\epsilon_2} dt\, U_0(t)\, e^{\iota z v(t)}, \tag{24}$$

where t is now a continuous variable, and

$$U_0 = \frac{uv'}{2 \sin \frac{1}{2}v'}\ e^{-\iota v'}, \tag{25}$$

the accent denoting differentiation with respect to t. The limits (ϵ_1, ϵ_2) are $(n_1, n_2+1)/z$ or, in the present case, $\frac{3}{2}z$ and ∞, which, for the present purpose, become zero and infinity.

Integrals of this type, containing a rapidly oscillating function, have already received attention. It may be shewn that, if at any point in the range, defined by suffix zero, it should happen that the derivate v' of the exponent becomes zero, the contributions to the integral in this neighbourhood are of a higher order of magnitude, and the value may be derived from the expression†

$$S = 2U_0 \left(\frac{\pi z}{2 \mid v_0'' \mid}\right)^{\frac{1}{2}} e^{\iota z v_0 \pm \frac{1}{4}\pi}, \tag{26}$$

the ambiguity depending upon whether v_0'' is positive or negative.

In the special case of the integrals between zero and infinity occurring in physical problems of the present type, the writer has carried the calculation to a higher order in the form of a series in descending powers of z. But this series is not required for the present problem, and therefore is not given now.

In a case in which the *forms* of (u, v) change in the range, a difficulty may arise owing to the increased importance of other terms for which v' is not nearly zero. This occurs, if at all, in the case of terms for which m and z are nearly equal, so that $R_n \propto m^{\frac{1}{3}}$. But, by writing a maximum value for each term of this kind, it is at once seen that this difficulty is not a feature of the present problem. The total number of such terms is of lower order than $m^{\frac{1}{3}}$. In fact, in all problems of this

* *Messenger of Maths.*, October, 1907.

† Lorenz, *Œuvres Scientifiques*, I, pp. 435 *et seq.*; Lord Kelvin, *Math. and Phys. Papers.*

nature, when there *is* a vanishing derivate of an exponent, it supplies the important part of the sum,* except, perhaps, when the vanishing occurs at $t = 0$, thus making U_0 of very low order.

Proceeding now to the summation of the harmonic series, it is found convenient to treat S_3 apart. In S_1 and S_2 the four exponents are given by (22), and their derivates become proportional to $2\partial\phi_n/z\partial t + \pi \pm \theta$ and $\pi \pm \theta$ respectively. The last two can never be zero, as points near $\theta = \pi$ are excluded. Moreover, as $\pi \pm \theta$ cannot be small, the first two can only vanish when $\partial\phi_n/z\partial t$ is of order unity, and therefore in the region of asymptotic expansions in which

$$\phi_n = \tfrac{1}{4}\pi + z\{(1-t^2)^{\frac{1}{2}} + t\sin^{-1}t - \tfrac{1}{2}\pi t\} \quad \text{or} \quad \frac{\partial\phi_n}{z\partial t} = 2\sin^{-1}t - \pi.$$

The possible points of zero derivate, henceforth called "zero points," must therefore satisfy

$$2\sin^{-1}t \pm \theta = 0,$$

and, as t is necessarily positive, only the second can do so. Its zero point is given by

$$t = \sin\tfrac{1}{2}\theta. \tag{27}$$

Thus the sums of the three series into which (S_1, S_2, S_3) can be decomposed are of a lower order of magnitude than that of the other, which contains $\iota z v_1$ as its exponent. Moreover, its sum is determined to the order we seek by the zero point $x = \sin\frac{1}{2}\theta$. Accordingly the important harmonics are such that m or $n+\frac{1}{2}$ is nearly equal to $\sin\frac{1}{2}\theta$, and, except for very small values of θ, the use of the expansion for the derivate of the zonal harmonic is justified.

Moreover, χ_n may be ignored from this point owing to the position of the significant harmonics, for which χ_n is of order z^{-1}. The rapid evanescence of harmonic terms for which n increases beyond z may be readily established independently of the above, and is, indeed, sufficiently obvious from general considerations. The summation of the harmonic series may now be effected.

Let a stand for (x, y, z), and let the operation

$$\left(-a\frac{2m}{k^2r^3} + \frac{2\iota m}{kn.n+1}\frac{\partial}{\partial a}\right)\tfrac{1}{4}\iota\sin\phi\left(\frac{2m}{\pi\sin\theta}\right)^{\frac{1}{2}} e^{-\iota kr+\frac{1}{2}\iota\pi}F, \tag{28}$$

acting on any function F, be denoted by $u.F$.

* The short investigation by Macdonald, *Proc. Roy. Soc.*, 1904, of the shadow cast by the earth indicates this physically, and is practically general in this respect. That the sum of such terms is really exponential, appears in a paper by the writer in *Phil. Mag.*, July, 1910. See also Poincaré, *Rendiconti del Circ. Mat. di Palermo*, Marzo–Aprile, 1910.

Then the sum of the first two harmonic series becomes

$$S_1+S_2=\sum_{n=1}^{\infty} u.e^{\iota zv},$$

where, in the region of importance, zv now denotes, if $m/z=t$,

$$zv=2z\{(1-t^2)^{\frac{1}{2}}+t\sin^{-1}t-\tfrac{1}{2}\pi t\}-z\theta t+z\pi t, \tag{29}$$

after a little reduction, in which the series whose sum is of lower order in z have not been retained.

u may be regarded as an operation acting on the exponential in its own term. This consideration greatly shortens the analysis, and is readily seen to be legitimate.

Now, at the zero point, $t=\sin\frac{1}{2}\theta$. Thus

$$v_0=2\cos\tfrac{1}{2}\theta, \qquad v_0'=0, \qquad v_0''=2(1-t^2)^{-\frac{1}{2}}=2\sec\tfrac{1}{2}\theta,$$

and is always positive.

Thus the series has a sum

$$2U_{00}\left(\frac{\pi z}{2v_0''}\right)^{\frac{1}{2}}e^{\iota zv_0+\frac{1}{4}\iota\pi}, \tag{30}$$

selecting the proper sign in the ambiguity of (26). Moreover, since $v_0'=0$, U_{00} reduces to u_0 by (25), and thus, with a neglect of relative order z^{-1},

$$U_{00}.F=\left(-\frac{2az}{k^2r^3}\sin\tfrac{1}{2}\theta+\frac{2\iota}{kz}\operatorname{cosec}\tfrac{1}{2}\theta\,\frac{\partial}{\partial a}\right)\left(\frac{2z\sin\frac{1}{2}\theta}{\pi\sin\theta}\right)^{\frac{1}{2}}\tfrac{1}{4}\iota\sin\phi\,e^{-\iota kr+\frac{1}{4}\iota\pi},$$

and the sum becomes on reduction, remembering again that the operation $\partial/\partial a$ acts only on θ and ϕ, and that $z\equiv ka$,

$$S_1+S_2=\left(\frac{aa^2\sin\frac{1}{2}\theta}{2r^3}-\frac{\iota}{2k}\operatorname{cosec}\tfrac{1}{2}\theta\,\frac{\partial}{\partial a}\right)\sin\phi\,e^{-\iota kr+2\iota ka\cos\frac{1}{2}\theta}. \tag{31}$$

Now
$$\frac{\partial}{\partial a}=\frac{\partial\theta}{\partial a}\frac{\partial}{\partial\theta}+\frac{\partial\phi}{\partial a}\frac{\partial}{\partial\phi},$$

and the operation $\partial/\partial\theta$ acting on $\sin\phi\,e^{2\iota ka\cos\frac{1}{2}\theta}$ raises its order in ka, whereas $\partial/\partial\phi$ does not.

Thus neglecting $\partial/\partial\phi$, except near $\phi=0$, when it becomes important, and writing, to the first order, $\partial/\partial\theta\equiv-\iota ka\sin\frac{1}{2}\theta$, then

$$\frac{\partial}{\partial a}\equiv-\iota ka\sin\tfrac{1}{2}\theta\,\frac{\partial\theta}{\partial a}, \tag{32}$$

and, finally,

$$S_1+S_2=-\tfrac{1}{2}\sin\phi\left(\frac{\iota aa^2}{r^3}\sin\tfrac{1}{2}\theta+a\,\frac{\partial\theta}{\partial a}\right)e^{-\iota kr+2\iota ka\cos\frac{1}{2}\theta}. \tag{33}$$

Summation of S_3.—This follows very quickly by a similar analysis. Retaining only that portion of S_3 which contains an exponent whose derivate can vanish, it may be written in the form

$$S_3 = \sum_{n=1}^{\infty} u . e^{\iota z v},$$

where zv again has the value of (29) and u is now an operation whose action on a function F is defined by

$$uF \equiv -\frac{m}{2krn.n+1}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \ \ldots, \ \ldots\right)\left(\frac{2m}{\pi \sin\theta}\right)^{\frac{1}{2}} \cos\phi\, e^{\frac{1}{2}\iota\pi - \iota kr} F. \quad (34)$$

Thus the zero point being unaltered,

$$U_{00} = u_0 = -\frac{1}{2kr}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \ \ldots, \ \ldots\right) \cos\phi \operatorname{cosec}\tfrac{1}{2}\theta (\pi ka \cos\tfrac{1}{2}\theta)^{-\frac{1}{2}}\, e^{\frac{1}{2}\iota\pi - \iota kr},$$

and, on reduction with the usual summation formula,

$$S_3 = -\frac{\iota}{2kr}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \ \ldots, \ \ldots\right) \cos\phi \operatorname{cosec}\tfrac{1}{2}\theta\, e^{-\iota kr + 2\iota ka\cos\frac{1}{2}\theta}. \quad (35)$$

Again, differentiation with respect to θ raises the order by ka, but not differentiation with respect to ϕ. The sum is of lower order in ka when $\phi = \frac{1}{2}\pi$, just as is S_1+S_2 when $\phi = 0$. But the sum $S_1+S_2+S_3$ is only required to the first order, and therefore this point is not important.

We may therefore write

$$\left(y\frac{\partial}{\partial x} - z\frac{\partial}{\partial y}, \ \ldots, \ \ldots\right) \equiv -\iota ka \sin\tfrac{1}{2}\theta \left(y\frac{\partial\theta}{\partial z} - z\frac{\partial\theta}{\partial y}, \ \ldots, \ \ldots\right).$$

But it may be shewn that

$$\frac{\partial\theta}{\partial x} = -\frac{\sin\theta}{r}, \qquad \frac{\partial\theta}{\partial y} = \frac{\cos\theta\cos\phi}{r}, \qquad \frac{\partial\theta}{\partial z} = \frac{\cos\theta\sin\phi}{r},$$

so that
$$\left(y\frac{\partial\theta}{\partial z} - z\frac{\partial\theta}{\partial y}, \ \ldots, \ \ldots\right) = (0, -\sin\phi, \cos\phi), \quad (36)$$

and thus
$$S_3 = -\frac{a\cos\phi}{2r}(0, -\sin\phi, \cos\phi)\, e^{-\iota kr + 2\iota ka\cos\frac{1}{2}\theta}. \quad (37)$$

Final Results.—In the value of S_1+S_2, the term in r^{-3} may be ignored, as this order of magnitude was neglected at an early stage. Thus

$$S_1+S_2 = -\tfrac{1}{2}a\sin\phi\,\frac{\partial\theta}{\partial a}\, e^{-\iota kr + 2\iota ka\cos\frac{1}{2}\theta},$$

where a denotes (x, y, z) or $r(\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi)$, from which

may be deduced $$r\frac{\partial\theta}{\partial\alpha} = (-\sin\theta,\ \cos\theta\cos\phi,\ \cos\theta\sin\phi), \tag{38}$$

and thus

$$S_1+S_2 = -\frac{a}{2r}\sin\phi\,(-\sin\theta,\ \cos\theta\cos\phi,\ \cos\theta\sin\phi)\,e^{-\iota kr+2\iota ka\cos\frac{1}{2}\theta}. \tag{39}$$

But $$(f,\ g,\ h) = S_1+S_2+S_3,$$

and, finally,
$$\left.\begin{aligned} f &= \frac{a}{2r}\sin\phi\sin\theta\,e^{-\iota kr+2\iota ka\cos\frac{1}{2}\theta},\\ g &= \frac{a}{2r}\sin\phi\cos\phi\,(1-\cos\theta)\,e^{-\iota kr+2\iota ka\cos\frac{1}{2}\theta},\\ h &= -\frac{a}{2r}\,[1-(1-\cos\theta)\sin^2\phi]\,e^{-\iota kr+2\iota ka\cos\frac{1}{2}\theta}, \end{aligned}\right\} \tag{40}$$

giving the distant components of electric polarization radiated from the sphere corresponding to an incident polarization

$$f_0 = 0, \qquad g_0 = 0, \qquad h_0 = e^{\iota kx}$$

along the negative direction of the axis of x. Since

$$\frac{4r^2\,|f^2+g^2+h^2\,|}{a^2}$$

$$= \sin^2\theta\sin^2\phi+\sin^2\phi\cos^2\phi\,(1-\cos\theta)^2+[1-(1-\cos\theta)\sin^2\phi]^2 = 1,$$

identically, we deduce $$|f^2+g^2+h^2\,| = \frac{a^2}{4r^2}, \tag{41}$$

where the square of a time factor has been neglected. Thus the mean energy in a volume element of space at distance r, of the scattered light, follows a very simple law. This, in fact, constitutes a verification of the results. For the whole energy radiated off by the sphere may be deduced at once from the energy stopped by the sphere when the latter is large, and the result is immediately seen to be in accord with this formula.

In terms of Cartesian coordinates,

$$(f,\ g,\ h) = \frac{a}{2r^2}\left\{z,\ \frac{yz(r-x)}{y^2+z^2},\ -\frac{ry^2+xz^2}{y^2+z^2}\right\}e^{-\iota kr+2\iota ka\cos\frac{1}{2}\theta}. \tag{42}$$

So far as the present investigation has gone, the results might have been obtained by the method of geometrical optics, as the sphere is large and perfectly reflecting, and it is a matter for surprise that this investigation has not hitherto been made. But the analysis given in this paper is useful in that it may be extended to give higher approximations without

much alteration, and these the writer will deal with in a second paper. Meanwhile, the present results are of importance in many ways, and more particularly in the examination of the size of the particles in the tails of comets.

As already indicated, the dependence of the intensity on the wave length is not to be expected. The results, as belonging to the other extreme case, contrast with Lord Rayleigh's and other investigators' formulæ for the small obstacle. They have not been proved to hold close to the axis. On the shadowed side, the significant harmonics would be in the neighbourhood of $m = ka$, and the corresponding effect of a different order of magnitude. But this is only the case in a very limited region. This region is such that the value of m given by $m = ka\cos\frac{1}{2}\theta$ makes $m-ka$ not of higher order than $ka^{\frac{1}{3}}$ or, practically, θ of order $ka^{-\frac{1}{3}}$. For visible light and a sphere of an inch diameter, $ka = 1{\cdot}4.10^6$. The limitations are therefore well defined, and, in the present case, θ does not exceed about two degrees.

For points very close to the axis on the unshadowed side, the zonal harmonic expansion cannot be used. But, by the use of the Mehler-Dirichlet integral formula for the zonal harmonic, it appears that the order of the effect does not change. This investigation is at present reserved.

We notice that

$$fx+gy+hz = 0, \tag{43}$$

so that the scattered wave becomes spherical at a great distance.

Comparison with the Optical Method.—As stated already, since the reflection is total, the first order approximation may be derived by the method of geometrical optics. A brief formal reconciliation of the results of the two methods is now given. The direction of vibration and the phase of the scattered wave are alone to be examined, for the results in other respects have already been shewn to be identical.

In order to be reflected along the direction (θ, ϕ), a ray incident parallel to the axis of x must have an angle of incidence equal to $\frac{1}{2}\theta$, and be incident at the point

$$(x_1 y_1 z_1) \equiv (a\cos\tfrac{1}{2}\theta,\ a\sin\tfrac{1}{2}\theta\cos\phi,\ a\sin\tfrac{1}{2}\theta\sin\phi), \tag{44}$$

and, if R be the distance between this point and the distant point (x, y, z) on the reflected ray, the ray which is incident, containing the factor

$$e^{\iota k x_1} = e^{\iota k a\cos\frac{1}{2}\theta}$$

at the surface, must finally contain $\dfrac{e^{\iota k x_1 - \iota k R}}{R}$,

from which the phase may be deduced. Now

$$R^2 = (r\cos\theta - a\cos\tfrac{1}{2}\theta)^2 + (r\sin\theta\cos\phi - a\sin\tfrac{1}{2}\theta\cos\phi)^2 + (r\sin\theta\sin\phi - a\sin\tfrac{1}{2}\theta\sin\phi)^2$$
$$= r^2 - 2ra\cos\tfrac{1}{2}\theta + a^2,$$

so that, when r is large, neglecting a^2/r,

$$R = r - a\cos\tfrac{1}{2}\theta,$$

and
$$e^{ikx_1 - ikR} = e^{-ikr + 2ika\cos\frac{1}{2}\theta}, \tag{45}$$

which is the exponential obtained by the summation of the electrical series. The phases are therefore in accordance in the two cases.

Again, under the same conditions, the following scheme of direction cosines may be deduced readily :—

For the ray incident at $(\frac{1}{2}\theta, \phi)$,

$$(l, m, n) = (1, 0, 0).$$

For the normal to the plane of incidence,

$$(l, m, n) = (0, \sin\phi, -\cos\phi).$$

For the direction of the component of incident polarization in the plane of incidence,
$$(l, m, n) = (0, \cos\phi, \sin\phi).$$

For the corresponding direction of this component in the reflected ray,

$$(l, m, n) = (-\sin\theta, \cos\theta\sin\phi, \cos\theta\cos\phi).$$

Thus, the incident wave being given by

$$(f_0, g_0, h_0) = (0, 0, 1)e^{ikx},$$

it follows that the component perpendicular to the plane of incidence, before incidence, has an amplitude $-\cos\phi$, and, after reflection, its contributions to the Cartesian components have amplitudes

$$(0, \sin\phi\cos\phi, -\cos^2\phi). \tag{46}$$

The corresponding contributions for the component vibrating in the plane of incidence become

$$(\sin\theta\sin\phi, -\cos\theta\sin\phi\cos\phi, -\cos\theta\sin^2\phi). \tag{47}$$

These contributions, when the wave scattered by the obstacle has proceeded to a distance, will follow the same law as those derived from the vibration normal to the plane of incidence, and their mode of dependence on the angular coordinates will not change. The factor of distance may be derived at once from the energy equation and becomes

$a/2r$. Thus the final Cartesian components, introducing the phase as calculated earlier, become, on summation,

$$\left.\begin{aligned} f &= \frac{a}{2r}\sin\phi\sin\theta\, e^{-ikr+2ika\cos\frac{1}{2}\theta}, \\ g &= \frac{a}{2r}\sin\phi\cos\phi\,(1-\cos\theta)\, e^{-ikr+2ika\cos\frac{1}{2}\theta}, \\ h &= -\frac{a}{2r}\left[1-(1-\cos\theta)\sin^2\phi\right] e^{-ikr+2ika\cos\frac{1}{2}\theta}, \end{aligned}\right\} \qquad (48)$$

and are identical with (40). The two methods thus give the same result to a first approximation. The second approximation, which is necessarily calculated by the electrical method, will be given in another paper, as it involves some extension of the analysis.

TERNARY QUADRATIC TYPES

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THE superior limit to the number of irreducible forms associated with five or less ternary quadratics.

In particular, a tabulated list of forms which make the complete system for four conics in a plane.

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[I use the word "type" in connection with five or less quadratics to mean a group of forms which are deducible from one another by interchanging all the symbols of any one quadratic of the system with all those of any other. Thus the point equations of the conics are of the same type; so are those of the Φ conics belonging to any two of the original conics, &c. However, in the first and last paragraphs, the word has the usual meaning defined in Grace and Young, Ch. xv, and used in Peano's Theorem, p. 358.]

1. The complete system of irreducible concomitants of two ternary quadratics has been found by Gordan* by a method which Ciamberlini† subsequently employed with success in the case of three quadratics. I

* Grace and Young, *Algebra of Invariants*, Ch. xiii.

† Ciamberlini, *Battaglini*, Vol. XXIV.

propose to use an extension of this method and establish a system for four and five quadratics. By doing so, the problem of finding the number of possible types of concomitants of any number of ternary quadratics is also solved; for Peano* has proved that this number is included in the types which arise for five or less conics together with the six-rowed determinant linear in the coefficients of six conics.

General Properties proved by Gordan† and holding for any Number of Quadratics.

2. Let the conics be called

$$f_0 = a_x^2 = b_x^2 = \ldots = 0_x^2, \qquad \Sigma_0 = u_\alpha^2 = u_\beta^2 = (uab)^2 = u_0^2,$$
$$f_1 = a_x'^2 = b_x'^2 = \ldots = 1_x^2, \qquad \Sigma_1 = u_{\alpha'}^2 = u_{\beta'}^2 = \ldots = u_1^2,$$
$$f_2 = a_x''^2 = b_x''^2 = \ldots = 2_x^2, \qquad \ldots,$$
$$\ldots,$$

where f and Σ are the usual correlative or reciprocal quadratic forms.

(I) Any concomitant P of these forms is composed of elements of the types

$$a_x, a_x', \ldots, u_\alpha, u_{\alpha'}, \ldots, (aa'u), \ldots, (aa'a''), \ldots, (a\alpha'u), \ldots, (a\alpha'\alpha''), \ldots, a_{\alpha'}, \ldots,$$

and as in the case of two conics, if there is a bracket (abp), we may transform the term to one containing two brackets (abp), (abq), and write a for (ab). Thus each symbol $a, a', \ldots, \alpha, \alpha', \ldots, b, b', \ldots,$ occurring in a form comes just twice.

(II) If P contains a factor $a_\alpha, a_{\alpha'}', \ldots,$ it is reducible.

(III) All forms obtained by permuting a, a, b, b in $a_p a_q b_r b_s M$ in all possible ways are equivalent. That is to say, the difference between any two of them is equal to a sum of terms in which the four symbols a, a, b, b have been replaced by two symbols α, α, and no other symbols have been altered in value. Thus the difference has been reduced.

The correlative holds for $\pi_\alpha \rho_\alpha \sigma_\beta \tau_\beta M$.

(IV) If a form contain two factors whose symbols are equivalent, it is reducible.

Such factors are $(aa'a'')(bb'b'')$, $(aa'u)(bb'u)$, $a_{\alpha'} b_{\beta'}, a_x b_x, u_\alpha u_\beta$, and so on.

* Grace and Young, *Algebra of Invariants*, p. 358.

† *Ibid.*, Ch. xiii.

By what has preceded,

$$(aa'a'')(bb'b'')M \equiv (aa'a'')(ab'b'')M_1 \equiv (aa'a'')(aa'b'')M_2 \equiv (aa'a'')(aa'a'')M_3,$$

which is reducible. The sign $\equiv$ expresses *equivalence*, so that $P \equiv Q$ implies $P-Q$ is reducible.

(V) If a form contains brackets of the types (pqr) and $(\pi\rho\sigma)$, where p, q, r are of type a or u, and π, ρ, σ are a or x, it is expressible in terms of brackets of one type, either (pqr) or $(\pi\rho\sigma)$, and factors p_π, for

$$(pqr)(\pi\rho\sigma) \equiv \Sigma p_\pi . q_\rho . r_\sigma,$$

on multiplying the two determinants together.

Change of Notation.

3. If $i, j, k, \ldots$ denote 0, 1, 2, ... in some order, we may write i for each symbol $a^i, b^i, \ldots$, which occurs; i for each symbol a^i which occurs in a conjunction p_{a^i}; $\bar{i}$ for each symbol a^i which occurs in a bracket.

Thus we write a term

$$(aa'a'')(aa'u)(a''bu)\, b_{a''} b'_{a''} b'_a b''_a b''_x,$$

as

$$(012)(01u)(20u)\, 0_2 1_2 1_0 2_0 2_x,$$

which is further abbreviated to

$$(012)(01u)(20u) \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & x \end{pmatrix}, \text{ or else to } \ldots (0_2 1_0 2_x). \tag{1}$$

This latter form stands for several different forms depending on the way the sets like a, a, b, b are distributed. But the preceding remarks shew that these forms are equivalent. Thus the numerical notation of a term includes groups of equivalent forms in one.

The correlative of form (1) is

$$(\overline{012})(\overline{01}x)(\overline{20}x) \begin{pmatrix} 2 & 0 & u \\ 0 & 1 & 2 \end{pmatrix}.$$

"Chains" and "Tags."

4. It is now evident that, if we consider all forms of type

$$P . Q,$$

where P is made up of bracket factors (012), (01u), ..., and Q of i_j, i_x, u_j, these terms and their correlatives form the whole set necessary for consideration.

Since P contains no symbols α, α', ..., any of the latter which occur must come twice in Q, so that Q is composed of what Ciamberlini calls "chains" and "knots."

A chain is $\begin{pmatrix} i & & i_1 & & j \\ & j_1 & & j_2 \cdots & \end{pmatrix}$, or (i, j), where two symbols i, j (*i.e.*, of type α) are connected by a continuous set of doublets. Written in full it is

$$\begin{matrix} \alpha^i & & \alpha^{i_1} & & \alpha^{j_1} \\ & \alpha^{j_1} & & \alpha^{j_2} & \cdots . \end{matrix}$$

A "knot" or "tag" is $\begin{pmatrix} i & & i_2 \cdots & & u \\ & j_1 & & j_n & \end{pmatrix}$, or else $\begin{pmatrix} i & & \cdots & i_n \\ & j_1 & & & x \end{pmatrix}$, written (i) for short. It is complete except for one odd symbol i.

The symbols i_1, ..., of type α, are the upper links, whereas j_1, ..., of type α, are called the lower links.

All the links i_1, ... must differ; for, if $i_r = s_s$, the form either contains the invariant $\begin{pmatrix} i_r & & \cdots & & i_s \\ & j_r & & j_{s-1} & \end{pmatrix}$, or is equivalent to a form containing this invariant.

Similarly, *all the lower links must differ.*

Again, *no two consecutive links i, j must be identical*, otherwise a factor like a_a occurs.

Method of Procedure.

5. We consider forms in the following order.

(1) Those with no bracket factors.

(2) Those with bracket factors.

I. *Forms with no bracket factors.*

It is obvious that these forms must be either (i, i) or $(i)(i)$ in character. For the form must be composed of chains and tags, and each initial and final figure in a chain or tag must occur elsewhere in the form.

(1) Consider the type (i, i).

For four conics the possibilities are

$$\begin{pmatrix} 0 & 2 & 0 \\ 1 & 3 & \end{pmatrix}, \qquad \begin{pmatrix} 0 & 2 & 3 & 0 \\ 1 & 0 & 2 & \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 2 & 3 & 0 \\ 3 & 0 & 1 & 2 & \end{pmatrix},$$

and forms derived from these by interchanging 0, 1, 2, 3 amongst themselves. For no two upper links, lower links, or consecutive links must be the same (§ 4).

We observe that a chain can be written in many ways: in fact,

$$\begin{pmatrix} 0 & 2 & 0 \\ 1 & 3 & \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ 3 & 1 & \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 \\ 1 & 3 & \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 \\ 3 & 1 & \end{pmatrix}.$$

Definition.—The number of pairs of symbols employed in a form (i, i) or $(i)(i)$ is called the *grade.*

The twelve forms of grade four,

$$\begin{pmatrix} 0 & 2 & 0 \\ 1 & 3 & \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 2 & 3 & \end{pmatrix}, \quad \ldots,$$

are irreducible. For five conics there are sixty of these forms.

The next type $\begin{pmatrix} 0 & 2 & 3 & 0 \\ 1 & 0 & 2 & \end{pmatrix}$ is of the sixth grade.

We have the identity

$$\begin{vmatrix} 0_0 & 1_0 & 2_0 \\ 0_1 & 1_1 & 2_1 \\ 0_3 & 1_3 & 2_3 \end{vmatrix} \equiv 0_1.1_3.2_0 - 0_1.1_0.2_3 + 0_3.1_0.2_1 \bmod (0_0, 1_1),$$

using the modular notation.

Squaring both sides of this, we have

$$\tfrac{1}{2}(012)^2(\overline{013})^2 \equiv 0_1.1_3.2_0.0_3.1_0.2_1 \bmod (\text{reducible terms}).$$

Thus $\begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & 0 & 3 & \end{pmatrix}$ is reducible, or as we shall say

$$\begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & 0 & 3 & \end{pmatrix} \equiv 0.$$

There remains the type of grade 8, $\begin{pmatrix} 0 & 1 & 2 & 3 & 0 \\ 3 & 0 & 1 & 2 & \end{pmatrix}$. This will be proved reducible later.

6. (2) Consider $(i)(i)$. Such forms are of three kinds,

$$\begin{pmatrix} u & i_1 & \ldots & u \\ j_1 & & & \end{pmatrix}, \quad \begin{pmatrix} u & i_1 & \ldots & i_k \\ j_1 & & & x \end{pmatrix}, \quad \begin{pmatrix} i_1 & & \ldots & i_k \\ x & j_1 & & x \end{pmatrix}.$$

Of these the first and third are correlative. Let the forms be taken in ascending grade G.

For grade less than four, the number of conics represented is less than four.

Grade 4.— $\begin{pmatrix} u & 1 & 3 \\ 0 & 2 & x \end{pmatrix}$. (1)

There are twenty-four of these for four conics, all irreducible and unconnected. For five conics the number is 5!.

Grade 5.—We have the covariants of three types

$$I_1 = \begin{pmatrix} 0 & 1 & 2 \\ x & 2 & 0 & x \end{pmatrix}, \quad I_2 = \begin{pmatrix} 0 & 1 & 2 \\ x & 3 & 0 & x \end{pmatrix}, \quad I_3 = \begin{pmatrix} 0 & 1 & 2 \\ x & 3 & 4 & x \end{pmatrix}.$$

The first is reduced by Ciamberlini, using $\{(012)(\overline{x20})\}^2$ as above. The last leads to $\{5!/2\}$ forms for five conics.

As for the second, we have

$$(012)(\overline{x30}) \equiv 0_x(1_3.2_0 - 1_0.2_3) + 0_3(1_0.2_x - 1_x.2_0) \bmod 0_0.$$

Squaring both sides,

$$\begin{pmatrix} 0 & 1 & 2 \\ x & 3 & 0 & x \end{pmatrix} + \begin{pmatrix} 0 & 2 & 1 \\ x & 3 & 0 & x \end{pmatrix} \equiv 0,$$

so that the symbols 1, 2 are practically interchangeable in the upper row.

For four conics the number of these forms is 24, so we take 12 for the irreducible system.

The rest of the forms for $G = 5$ are correlative.

7. *Grade* 6.—The general form is $\begin{pmatrix} i & j & k & u \\ x & l & m & n \end{pmatrix}$. We shall shew that $\begin{pmatrix} i & j & k & u \\ x & l & i & j \end{pmatrix}$ is reducible.

Four Conics.—Let i, j, k be 0, 1, 2 in some order, and l, m, n, be 0, 1, 3. Suppose that

$$\left.\begin{aligned} L &= \begin{pmatrix} 0 & 1 & 2 & u \\ x & 3 & 0 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 & 2 & u \\ x & 3 & 1 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} 2 & 1 & 0 & u \\ x & 0 & 3 & 1 \end{pmatrix} \\ L' &= \begin{pmatrix} 0 & 2 & 1 & u \\ x & 1 & 0 & 3 \end{pmatrix}, \quad M' = \begin{pmatrix} 1 & 2 & 0 & u \\ x & 0 & 1 & 3 \end{pmatrix}, \quad N' = \begin{pmatrix} 2 & 0 & 1 & u \\ x & 1 & 3 & 0 \end{pmatrix} \\ L'' &= \begin{pmatrix} 0 & 2 & 1 & u \\ x & 1 & 3 & 0 \end{pmatrix}, \quad M'' = \begin{pmatrix} 1 & 2 & 0 & u \\ x & 0 & 3 & 1 \end{pmatrix} \end{aligned}\right\}. \quad (\alpha)$$

These are all the possible forms with 0, 1, 2 and 1, 0, 3 as upper and lower links, avoiding of course 0_0 or 1_1.

Since $$\begin{pmatrix} 0 & 2 & 1 \\ x & 1 & 0 & x \end{pmatrix} \equiv 0,$$

we may deduce $$L'+M' \equiv 0, \tag{1}$$

by polarizing with regard to (3^x), which is effected by differentiating the identity with respect to x and writing 3 for x and adding u to the top row.

From the identity

$$\begin{pmatrix} 0 & 1 & 2 \\ x & 3 & 0 & x \end{pmatrix} + \begin{pmatrix} 0 & 2 & 1 \\ x & 3 & 0 & x \end{pmatrix} \equiv 0,$$

we deduce

$$\begin{pmatrix} 0 & 1 & 2 & u \\ x & 3 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 0 & u \\ x & 0 & 3 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2 & 1 & u \\ x & 3 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 0 & u \\ x & 0 & 3 & 1 \end{pmatrix} \equiv 0,$$

or $$L+N+M'' \equiv 0. \tag{2}$$

Similarly $$L''+M+N' \equiv 0. \tag{3}$$

By polarizing the correlative forms, we obtain

$$N+N' \equiv 0, \tag{4}$$

$$L+L'+L'' \equiv 0, \tag{5}$$

$$M+M'+M'' \equiv 0. \tag{6}$$

These six relations are equivalent to the following five,

$$\left.\begin{aligned} L'' &\equiv -(L+L') \\ M &\equiv L+L'+N \\ M' &\equiv -L' \\ M'' &\equiv -(L+N) \\ N' &\equiv -N \end{aligned}\right\}. \tag{7}$$

But $$L'+M+N \equiv 0.$$

For $L'+M+N$ is

$$0_1.2_0.1_3\{0_x.2_1.1_0.u_3+1_x.0_3.2_1.u_0+2_x.1_0.0_3.u_1\}$$

$$\equiv 0_1.2_0.1_3\{0_x(21u)(\overline{103})+1_x(02u)(\overline{310})+2_x(10u)(\overline{031})\}, \tag{8}$$

neglecting reducible terms.

By a fundamental identity the terms (8) are equal to

$$0_1.2_0.1_3(\overline{103})(012)\,u_x \text{ which } \equiv 0.$$

Thus $$L'+M+N \equiv 0$$

Similarly $$L+M+N' \equiv 0 \tag{9}$$

Combining relations (7) and (9),

$$\left.\begin{aligned} L+2L'+2N &\equiv 0 \\ L-L'-N &\equiv 0 \end{aligned}\right\},$$

whence $L \equiv 0$, $M \equiv 0$, and all the rest are either $\equiv L'$ or $\equiv -L'$.

The forms L, M are of type

$$\begin{pmatrix} i & j & k & u \\ x & l & i & j \end{pmatrix}. \qquad (A)$$

This is reducible; and the other forms lead to twelve irreducible forms for four conics, twelve being the number of groupings $\begin{pmatrix} i & j & k \\ i & j & l \end{pmatrix}$ out of 0, 1, 2, 3, the order ijk or ijl being immaterial.

8. *Grade 6 for Five Conics.*—A similar reduction, not so important, holds for this case.

Take

$$\left.\begin{aligned} L_1 &= \begin{pmatrix} 0 & 1 & 2 & u \\ x & 3 & 0 & 4 \end{pmatrix}, & M_1 &= \begin{matrix} 0 & 1 & 2 \\ 3 & 4 & 0 \end{matrix}, & N_1 &= \begin{matrix} 1 & 0 & 2 \\ 3 & 4 & 0 \end{matrix}, & P_1 &= \begin{matrix} 1 & 2 & 0 \\ 0 & 3 & 4 \end{matrix} \\ L_2 &= \begin{matrix} 0 & 2 & 1 \\ 3 & 0 & 4 \end{matrix}, & M_2 &= \begin{matrix} 0 & 2 & 1 \\ 3 & 4 & 0 \end{matrix}, & N_2 &= \begin{matrix} 2 & 0 & 1 \\ 3 & 4 & 0 \end{matrix}, & P_2 &= \begin{matrix} 2 & 1 & 0 \\ 0 & 3 & 4 \end{matrix} \\ L_3 &= \begin{matrix} 0 & 1 & 2 \\ 4 & 0 & 3 \end{matrix}, & M_3 &= \begin{matrix} 0 & 1 & 2 \\ 4 & 3 & 0 \end{matrix}, & N_3 &= \begin{matrix} 1 & 0 & 2 \\ 4 & 3 & 0 \end{matrix}, & P_3 &= \begin{matrix} 1 & 2 & 0 \\ 0 & 4 & 3 \end{matrix} \\ L_4 &= \begin{matrix} 0 & 2 & 1 \\ 4 & 0 & 3 \end{matrix}, & M_4 &= \begin{matrix} 0 & 2 & 1 \\ 4 & 3 & 0 \end{matrix}, & N_4 &= \begin{matrix} 2 & 0 & 1 \\ 4 & 3 & 0 \end{matrix}, & P_4 &= \begin{matrix} 2 & 1 & 0 \\ 0 & 4 & 3 \end{matrix} \end{aligned}\right\} \qquad (\beta)$$

where x, u are understood to occur in each case.

Polarize $$\begin{pmatrix} 0 & 1 & 2 \\ x & 3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2 & 1 \\ x & 3 & 0 \end{pmatrix} \equiv 0,$$

with regard to (4^u) and we deduce

Similarly,
$$\left.\begin{aligned} L_1+L_2+P_1+P_2 &\equiv 0 \\ N_1+N_3+P_1+P_3 &\equiv 0 \\ L_3+L_4+P_3+P_4 &\equiv 0 \\ N_2+N_4+P_2+P_4 &\equiv 0 \end{aligned}\right\}. \qquad (\beta')$$

Further,

$$L_1+M_2+N_4+P_3 = 0_3.1_0.2_4\{0_x.1_3.2_0.u_4+0_x.2_3.1_4.u_0 + 2_x.0_4.1_3.u_0+1_x.2_0.0_4.u_3\} \equiv 0,$$

on using the expansion of the vanishing determinant

$$\begin{vmatrix} 0_x & 1_x & 2_x & u_x \\ 0_3 & 1_3 & 2_3 & u_3 \\ 0_0 & 1_0 & 2_0 & u_0 \\ 0_4 & 1_4 & 2_4 & u_4 \end{vmatrix}.$$

Thus
$$L_1+M_2+N_4+P_3 \equiv 0$$
Similarly,
$$\left.\begin{aligned} L_2+M_1+N_3+P_4 &\equiv 0 \\ L_3+M_4+N_2+P_1 &\equiv 0 \\ L_4+M_3+N_1+P_2 &\equiv 0 \end{aligned}\right\}. \qquad (\beta'')$$

These eight relations are independent and shew that eight of the sixteen forms $L_1 \ldots P_4$ may be taken for the irreducible system.

9. *Grade* 7.—*Four Conics.*—The type to consider is

$$\begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ x & j_1 & j_2 & j_3 & x \end{pmatrix}$$

and its correlative.

The i's must be 0, 1, 2, 3 in some order. Consider

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ x & j_1 & j_2 & j_3 & x \end{pmatrix},$$

where $j_1 j_2 j_3$ are 0, 1, 2 in some order, or 0, 1, 3 in some order. These include all cases.

If 0, 1, 2, then j_3 must be 1, j_1 must be 2, so that $j_2 = 0$. But

$$\begin{pmatrix} 0 & 1 & 2 & u \\ x & 2 & 0 & 1 \end{pmatrix} \equiv 0,^*$$

hence
$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ x & 2 & 0 & 1 & x \end{pmatrix} \equiv 0,$$

on polarizing with regard to $\left(3_x \frac{\partial}{\partial u}\right)$.

If the j's are 0, 1, 3, j_3 must be 1, j_2 must be 0, so the same argument applies.

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* Since $\begin{pmatrix} 0 & 1 & 3 & u \\ x & 2 & 0 & 1 \end{pmatrix} \equiv 0$, *a fortiori* $\begin{pmatrix} 0 & 1 & 2 & u \\ x & 2 & 0 & 1 \end{pmatrix} \equiv 0$.

Thus all forms for $G = 7$ are reducible.

By writing j_4 for x, we prove the invariant $\begin{pmatrix} 1 & 2 & 3 & 0 \\ 3 & 0 & 1 & 2 & 3 \end{pmatrix}$ reducible (§ 7, A).

10. *Grade* 8.—*Four Conics.*—The only forms are $\begin{pmatrix} 0 & 1 & 2 & 3 & u \\ x & p & q & r & s \end{pmatrix}$, where p, q, r, s are 0, 1, 2, 3 in some order.

Either $s = 2$ or $p = 2$. If $s = 2$, r must be 1. If $p = 2$, q must be 3.

In each case we get a form

$$\begin{pmatrix} \dots & i & j & l & \dots \\ & k & i & j \end{pmatrix}, \qquad (\gamma)$$

which is reducible, since

$$\begin{pmatrix} i & j & l & u \\ x & k & i & j \end{pmatrix} \equiv 0,$$

and we may replace x, u by any symbols we like.

No higher grade exists for four conics, as we should have to repeat like links in the upper or lower row.

Summary of Results for Four Conics.

11. $G = 4$, $\begin{pmatrix} 0 & 1 & u \\ x & 2 & 3 \end{pmatrix}$, 24 forms,

$\begin{pmatrix} 0 & 1 & 0 \\ 2 & 3 \end{pmatrix}$, 12 forms.

$G = 5$, $\begin{pmatrix} 0 & 1 & 2 \\ x & 3 & 0 & x \end{pmatrix} \equiv -\begin{pmatrix} 0 & 2 & 1 \\ x & 3 & 0 & x \end{pmatrix}$, 12 forms,
12 correlative forms.

$G = 6$, $\begin{pmatrix} 0 & 1 & 2 & u \\ x & 2 & 0 & 3 \end{pmatrix}$, 12 forms, one for each set obtained by dropping out a pair of different symbols 3, 1 from the two sets (0, 1, 2, 3), (0, 1, 2, 3).

Case for Five Conics.

12. For $G = 6$, we must add $\begin{pmatrix} 0 & 1 & 2 & 0 \\ 2 & 3 & 4 \end{pmatrix}$ to those already found.

By writing 4 for x in the set I_2, we see that 0, 1 are interchangeable in this. Similarly 3, 4 are. There are $5\times\frac{4.3}{1.2}$ or 30 of these.

Grade 7.—The type is $\begin{pmatrix} i & j & k & l \\ x & m & n & p & x \end{pmatrix}$ or its correlative.

Here we must have one of the five, 4 say, missing in the top row, and two, 1, 2 say, missing from the lower row.

If in the sets (β'), (β'') of Grade 6, we write 3_y for u, we obtain the results

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ x & 3 & 0 & 4 & y \end{pmatrix}+\begin{pmatrix} 0 & 2 & 1 & 3 \\ x & 3 & 0 & 4 & y \end{pmatrix}\equiv 0,$$

$$\left.\begin{aligned} \begin{pmatrix} 1 & 0 & 2 & 3 \\ x & 3 & 4 & 0 & y \end{pmatrix}+\begin{pmatrix} 1 & 0 & 2 & 3 \\ x & 4 & 3 & 0 & y \end{pmatrix}&\equiv 0 \\ \begin{pmatrix} 0 & 1 & 2 & 3 \\ x & 3 & 4 & 0 & y \end{pmatrix}+\begin{pmatrix} 1 & 0 & 2 & 3 \\ x & 4 & 3 & 0 & y \end{pmatrix}&\equiv 0 \end{aligned}\right\},\ \text{from } (\beta'),$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ x & 4 & 3 & 0 & y \end{pmatrix}+\begin{pmatrix} 1 & 0 & 2 & 3 \\ x & 3 & 4 & 0 & y \end{pmatrix}\equiv 0,\quad \text{from } (\beta''),$$

neglecting the reducible types

$$\begin{pmatrix} \dots & i & j & \dots \\ & \dots & i & j \end{pmatrix}\quad\text{and}\quad \begin{pmatrix} \dots & & i & j & \dots \\ & \dots & i & j \end{pmatrix},$$

which occur.

Thus

$$\left.\begin{aligned} \begin{pmatrix} 1 & 0 & 2 & 3 \\ x & 3 & 4 & 0 & y \end{pmatrix}&\equiv -\begin{pmatrix} 1 & 0 & 2 & 3 \\ x & 4 & 3 & 0 & y \end{pmatrix}\equiv \begin{pmatrix} 0 & 1 & 2 & 3 \\ x & 3 & 4 & 0 & y \end{pmatrix} \\ &\equiv -\begin{pmatrix} 0 & 1 & 2 & 3 \\ x & 4 & 3 & 0 & y \end{pmatrix}. \end{aligned}\right.$$

Similarly,

$$\left.\begin{aligned} \equiv\begin{pmatrix} 0 & 1 & 3 & 2 \\ x & 3 & 4 & 0 & y \end{pmatrix}&\equiv -\begin{pmatrix} 0 & 1 & 3 & 2 \\ x & 3 & 0 & 4 & y \end{pmatrix}\equiv \begin{pmatrix} 0 & 1 & 2 & 3 \\ x & 3 & 0 & 4 & y \end{pmatrix}, \\ \text{and } \begin{pmatrix} 1 & 0 & 2 & 3 \\ x & 3 & 4 & 0 & y \end{pmatrix}&\equiv -\begin{pmatrix} 1 & 0 & 3 & 2 \\ x & 3 & 4 & 0 & y \end{pmatrix}\equiv \begin{pmatrix} 1 & 3 & 0 & 2 \\ x & 0 & 4 & 3 & y \end{pmatrix} \\ &\equiv -\begin{pmatrix} 1 & 3 & 2 & 0 \\ x & 0 & 4 & 3 & y \end{pmatrix} \end{aligned}\right\}, \qquad (\delta)$$

on using the results of § 7, about the equivalence of forms

$$\begin{pmatrix} i & k & j & u \\ x & j & l & i \end{pmatrix}, \quad \begin{pmatrix} k & i & j & u \\ x & j & l & i \end{pmatrix}, \quad \ldots.$$

Now the forms with 0, 1, 2, 3 at the top, and 0, 3, 4 at the bottom, not of type (γ), are

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ x & 3 & 4 & 0 & y \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 2 & 3 \\ x & 4 & 3 & 0 & y \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 2 & 3 \\ x & 3 & 0 & 4 & y \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 3 & 2 \\ x & 3 & 0 & 4 & y \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 3 & 2 \\ x & 3 & 4 & 0 & y \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 2 & 3 \\ x & 4 & 3 & 0 & y \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 2 & 3 \\ x & 3 & 4 & 0 & y \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 3 & 2 \\ x & 3 & 4 & 0 & y \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 0 & 2 \\ x & 0 & 4 & 3 & y \end{pmatrix},$$

$$\begin{pmatrix} 3 & 1 & 0 & 2 \\ x & 0 & 3 & 4 & y \end{pmatrix}, \quad \begin{pmatrix} 3 & 1 & 0 & 2 \\ x & 0 & 4 & 3 & y \end{pmatrix},$$

together with forms with 0 fourth in the top row, and those with 1 and 2 interchanged.

If x and y are identical, all these forms are equivalent, by (δ), except the second and third in the first row, but they are reducible if written as

$$\begin{pmatrix} 1 & 2 & \overbrace{3} & \overbrace{0} & 1 \\ \underbrace{3} & \underbrace{0} & x & 4 \end{pmatrix}, \quad \begin{pmatrix} 2 & \overbrace{3} & \overbrace{0} & 1 & 2 \\ 4 & x & \underbrace{3} & \underbrace{0} \end{pmatrix}, \quad \text{by } (\gamma).$$

For the above set is complete without interchanging 1 and 2, since we get this interchange by writing each term backwards.

It follows that each grouping (0 1 2 3, 0 3 4) *leads to one irreducible form of Grade* 7. *Thus we have* $5 \times \frac{4.3}{1.2}$ *or* 30 *new irreducible terms.*

13. *Grade* 8.—There are invariants $\begin{pmatrix} 0 & 1 & 2 & 3 & 0 \\ p & q & r & s \end{pmatrix}$ and forms $\begin{pmatrix} 0 & 1 & 2 & 3 & u \\ x & p & q & r & s \end{pmatrix}$.

The Invariants.—Let 0 be absent from the lower row (the case when 4 is absent is discussed under four conics). Then either $p = 2$ or $s = 2$. If $s = 2$, r must be 1, and the form is $\equiv 0$, by (γ). If $p = 2$, q must be 3 and the form is again reducible. Thus invariants of Grade 8 are reducible.

The type $\begin{pmatrix} 0 & 1 & 2 & 3 & u \\ x & p & q & r & s \end{pmatrix}$. Let 0 be absent from p, q, r, s. The form reduces as the previous invariant. Thus 0 is present in the bottom row. If 1 is absent from the lower row, we may have

$$\begin{pmatrix} 0 & 1 & 2 & 3 & u \\ x & 3 & 0 & 4 & 2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 2 & 3 & u \\ x & 3 & 4 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 2 & 3 & u \\ x & 4 & 3 & 0 & 2 \end{pmatrix},$$

and $$\begin{pmatrix} 0 & 1 & 2 & 3 & u \\ x & 2 & 3 & 4 & 0 \end{pmatrix} \equiv 0, \text{ by } (\gamma).$$

The results (δ) shew that 0, 1 may be interchanged in the top row in each of these; so they come under the previous case and are reducible.

If 2 is absent, we have

(1) $\begin{pmatrix} 0 & 1 & 2 & 3 & u \\ x & 3 & 0 & 4 & 1 \end{pmatrix}$, (2) $\begin{pmatrix} 0 & 1 & 2 & 3 & u \\ x & 4 & 3 & 0 & 1 \end{pmatrix}$, (3) $\begin{pmatrix} 0 & 1 & 2 & 3 & u \\ x & 3 & 4 & 0 & 1 \end{pmatrix}$,

(4) $\begin{pmatrix} 0 & 1 & 2 & 3 & u \\ x & 3 & 4 & 1 & 0 \end{pmatrix}$, (5) $\begin{pmatrix} 0 & 1 & 2 & 3 & u \\ x & 4 & 3 & 1 & 0 \end{pmatrix}$.

Here (1), (2), (3) are equivalent, by (δ). But 1, 4 are interchangeable in the bottom row of (1) by (δ'); hence $(1) \equiv 0$, by (γ). Similarly, $(3) \equiv (4)$ and $(4) \equiv (5)$ by (δ'), (δ') denoting the correlative identities to (δ) of type

$$\begin{pmatrix} u & 1 & 2 & 3 & v \\ 4 & 3 & 1 & 0 \end{pmatrix} \equiv \begin{pmatrix} u & 1 & 2 & 3 & v \\ 3 & 4 & 1 & 0 \end{pmatrix}.$$

Thus all (1) to (5) are $\equiv 0$.

Finally, if 3 is absent, we have

$$\begin{pmatrix} 0 & 1 & 2 & 3 & u \\ x & 2 & 4 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 2 & 3 & u \\ x & 2 & 0 & 4 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 2 & 3 & u \\ x & 2 & 4 & 0 & 1 \end{pmatrix}.$$

The formulæ (δ') reduce these to be equivalent to

$$\begin{pmatrix} 0 & 1 & 2 & 3 & u \\ x & 2 & 0 & 1 & 4 \end{pmatrix},$$

or a reducible form by (γ).

Hence all forms for $G = 8$ are reducible. Those for higher grades are deducible from

$$\begin{pmatrix} p & q & r & s & u \\ x & p_1 & q_1 & r_1 & s_1 \end{pmatrix},$$

by writing u_{p_0} for x or t_x for u, so they are reducible. Thus the possibilities are exhausted.

For five conics we have the system

$$G = 5, \quad \begin{pmatrix} 0 & 1 & 2 \\ x & 3 & 4 & x \end{pmatrix}, \quad \begin{pmatrix} u & 3 & 4 & u \\ 0 & 1 & 2 \end{pmatrix},$$

$$G = 6, \quad \begin{pmatrix} 0 & 1 & 2 & 0 \\ 2 & 3 & 4 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 2 & u \\ x & 3 & 0 & 4 \end{pmatrix},$$

$$G = 7, \quad \begin{pmatrix} 0 & 1 & 2 & 3 \\ x & 2 & 0 & 4 & x \end{pmatrix}, \text{ and correlatives.}$$

Identities and their Nomenclature.

14. The two principal identities are

(i) $(012)\,3_p - (013)\,2_p - (032)\,1_p - (312)\,0_p = 0,$

(ii) $(012)(034) + (023)(014) + (031)(024) = 0,$

which are written

(i) $\Sigma(\dot{0}\dot{1}\dot{2})\,3_p = 0,$

(ii) $\Sigma(\dot{0}\dot{1}\dot{2})(\dot{0}\dot{3}\dot{4}) = 0,$ or $\Sigma(\dot{0}\dot{1}\dot{2})(\dot{0}\dot{3}\dot{4}) = 0,$ a special case of (i).

The way they are used is generally expressible as follows:—

If the first bracket of (i), (ii) is called g_r, and if the symbol corresponding to 3 in (i) or (ii) belongs to a bracket g_s or an element 3_p, or a tag (3) we denote the identity by

$$(g_r,\ 3 \text{ of } 3_p) \quad \text{or} \quad [g_r, (\dot{3})].$$

Thus, in $(01u)(0_4\,2_0\,3_x)(1)$, we have

$$(01u)(0_4\,2_0\,3_x)(1) - (012)(0_4\,2)(u_0 3_x)(1) \ldots = 0,$$

and we say that

$$(01u)(0_4\,2_0\,3_x)(1) \equiv 0, \quad \text{by} \quad (g,\ \dot{2}_0).$$

Bracket factors are called $g_1 g_2 \ldots g_k$ in the order from left to right.

The System of Forms involving Bracket Factors.

15. We shall consider those only which have brackets (012) or $(01u)$. All others are correlatives of these, $(\overline{012})$, $(\overline{01}x)$.

The type of bracket (012) is a "full bracket," while the other sort, $(01u)$, is a "u-bracket."

If F denote a full bracket g, a u-bracket, a typical form is

$$F_1F_2 \dots F_m g_1 g_2 \dots g_n (i_1, i_2)(i_3, i_4) \dots (j_1)(j_2) \dots,$$

where all $i_1 i_2 \dots j_1 j_2$ differ, and have their complements in $F \dots g \dots$. We consider forms in ascending value of m, the number of full brackets.

The greatest number of full brackets is three.

In fact, consider the number of full brackets for five conics. It is impossible for more than six brackets to contain a symbol 0 say, without the same bracket recurring twice. With six such brackets we should have the form

$$(012)(034)(013)(024)(014)(023)\, M,$$

which is reducible as it contains the invariant

$$(012)(034)(013)(024).$$

Similarly, for five brackets containing 0.

In the case of four brackets we have either

$$(012)(034)(013)(024),$$

which is reducible by squaring the identity

$$(012)(034) - (013)(024) = (032)(014),$$

or

$$(012)(013)(014)(034)\, M,$$

which is reduced by $(\dot{g}_1, \dot{3}$ of $g_4)$.

The notation $(\dot{g}_1, \dot{3}$ of $g_4)$ means that the identity

$$(012)(034) - (032)(014) - (013)(024) = 0,$$

is used, g_r standing for the r-th bracket.

Thus, if the symbol 0 occur in four full brackets, the form is reducible. Further, if there are four full brackets and a symbol 0 occurs in three of them, there will be at least one more 0 in the form, which we take to be

$$(0ij)(0kl)(0mn)(pqr)\,0\ M.$$

By the identity $(\dot{g}_4, \dot{0}_s)$ we bring 0 into the fourth bracket. This gives the same case as before.

16. It follows that there are either less than four full brackets, or, if more than three, there is not more than one pair of any kind of like symbols in them—an obviously impossible case.

Thus we have the following types only:—

I.	M,	(§§ 17–22)
II.	$(012)\,M$,	(§§ 23–29)
III.	(*a*) $(012)(013)\,M$,	(§§ 30–32)
	(*b*) $(012)(034)\,M$,	(§§ 33–34)
IV.	(*a*) $(012)(013)(234)\,M$,	(§§ 35–38)
	(*b*) $(012)(013)(034)\,M$,	
	(*c*) $(012)(013)(014)\,M$,	
	(*d*) $(023)(031)(012)\,M$;	

where M denotes u-brackets and chains, &c.

We shall now examine these four main groups in detail.

I. *Forms with no Full Brackets.* (§§ 17–22.)

17. Consider forms with no full brackets. Let one such form P contain k u-brackets, so that $2k$ digits exist in these brackets, of which an even number may have complementary symbols outside the u-brackets. Either none, two, or four of such digits occur.

The possible types for the first case, in which P must contain u-brackets alone, are

(1) $(01u)^2$,

(2) $(01u)(12u)(20u)$,

(3) $(01u)(12u)(23u)(30u)$,

(4) $(01u)(12u)(23u)(34u)(40u)$.

For any other form must be made up of factors of these four kinds. Forms (1), (2) belong to less than four conics; whereas (3) and (4) are reducible. By squaring both sides of the identity

$$(01u)(23u)-(03u)(12u) \equiv (02u)(31u).$$

We reduce form (3).

By polarizing (3) with regard to $\left(a^{\mathrm{iv}}\frac{\partial}{\partial a}\right)$, we obtain the relation

$$(01u)(12u)(23u)(34u)+(41u)(12u)(23u)(30u) \equiv 0.$$

But $(01u)(12u)(23u)(34u)-(41u)(12u)(23u)(30u) = (31u)(12u)(23u)(04u)$.

Hence $(01u)(12u)(23u)(34u) \equiv \frac{1}{2}(31u)(12u)(23u)(04u)$.

Thus form (4) $\equiv 0 \bmod (04u)^2$ or is reducible.

Second Case.—Next, if two symbols are left, 0, 1 say, unpaired at the end of the u-brackets, the forms are

$$g_1 g_2 \dots g_k\ [0, 1] \quad \text{(where } [0, 1] \text{ denotes a chain or two tags),}$$

and must be of the following types

$$\left.\begin{array}{l}(01u)[0, 1]\\ (12u)(20u)[0, 1]\\ (12u)(23u)(30u)[0, 1]\end{array}\right\}, \qquad \text{I}$$

$$(12u)(23u)(34u)(40u)[0, 1],$$

the last being reducible by polarizing the reducible form

$$(12u)(23u)(34u)(41u)$$

as above.

Third Case.—Four symbols left over from the u-brackets. Exactly the same considerations shew that the possible kinds are

$$\left.\begin{array}{l}(03u)(12u)[0, 1][2, 3]\\ (04u)(21u)(34u)[0, 1][2, 3]\end{array}\right\}. \qquad \text{II}$$

If $[0, 1]$ is $(0)(1)$, we obtain a reduction to types I, by $[g_2, (0)]$. Thus there are only chains, and the possible types are

$$\left.\begin{array}{ll}(a) & (03u)(12u)(0_p 1)(2_q 3)\\ (b) & (03u)(12u)(0_p 4_q 1)(2_r 3)\\ (c) & (04u)(21u)(34u)(0_p 1)(2_q 3)\end{array}\right\} \quad \begin{array}{l}p, q, r \text{ being of type } i,\\ \textit{i.e.}, 0, 1, 2, 3, 4.\end{array}$$

(*b*) By $(g_2, 0) = (03u)(120)(u_p 4_q 1)(2_r 3) +$ terms of second case.

If $q = 0$, $= (01u)(120)(u_p 4_q 3)(2_r 3)$

$=$ second case, by (g_2, u).

So $q \neq 0$. Similarly $p \neq 1$. Therefore

$$q = 2 \text{ or } 3, \quad p = 3 \text{ or } 2.$$

But the form

$= (032)(12u)(0_p 4_q 1)(u_r 3) +$ second case

$= (031)(12u)(0_p 4_q 2)(u_r 3) +$ terms reducible to second case

$= (03u)(12u)(0_p 4_q 2)(1_r 3) +$ second case,

which is the same as (*b*) with 1, 2 interchanged; so p may not be 2. Obviously q may not be 2. Thus the form is always reducible to the second case.

As for $(03u)(12u)(0_p1)(2_q3)$. By the perpetuant reduction* we express this as a sum of terms with two similar elements in g_1g_2, and therefore as

$$\Sigma(01u)(012)(\ldots u),$$

$$\Sigma(01u)(02u)\ldots,$$

and $$\Sigma(012)(013)\ldots,$$

of type already considered. Of these the first is of the same type as the second, if (g_2, u) is used to bring both u's into g_1g_2.

We consider the third type (*c*)

$$(012)(013)(2_ku)(u_l3),$$

$k = 0, 1, 3$ or 4, not x; $l = 0, 1, 2$ or 4, not x.

If $k = 0$, by (g_1, u_k), this

$$\equiv (0u2)(013)(1_02)(u_l3)$$

$$\equiv (0u2)(012)(1_03)(u_l3) \quad [\text{by } (g_2, 2_0)]$$

$$\equiv \text{previous type by } (g_2, u_l).$$

Hence $k \neq 0$. Similarly, if $k = 1$.

Similarly, if $l = 0$ or 1.

But the form is also $\equiv (032)(013)(2_ku_l1)$

$$+ (312)(013)(2_ku_l0),$$

and these are reducible similarly if k, l contain 3. Similarly, too, k and l cannot be 2.

Thus as nothing is left but 4, the form is reducible in all cases.

So $(04u)(21u)(34u)(0_p1)(2_q3)$ is the only possibly irreducible form of this class.

18. The only forms left are now

$$\left.\begin{array}{l}(01u)[0, 1]\\(12u)(20u)[0, 1]\\(12u)(23u)(30u)[0, 1]\end{array}\right\}. \qquad \text{I}$$

* See § 34.

We divide these into (A) and (B):—(A) if [0, 1] is a chain, (B) if [0, 1] is of the form (0)(1).

The possibilities are

(A) (i) $(01u)\begin{pmatrix}0 & 2 & 3 & 4 & 1\\ p & q & r & s\end{pmatrix}$, (v) $(12u)(20u)\begin{pmatrix}0 & 3 & 4 & 1\\ p & q & r\end{pmatrix}$,

(ii) $(01u)\begin{pmatrix}0 & 2 & 3 & 1\\ p & q & r\end{pmatrix}$, (vi) $(12u)(20u)\begin{pmatrix}0 & 3 & 1\\ p & q\end{pmatrix}$,

(iii) $(01u)\begin{pmatrix}0 & 2 & 1\\ p & q\end{pmatrix}$, (vii) $(12u)(20u)\begin{pmatrix}0 & 1\\ 3\end{pmatrix}$,

(iv) $(01u)\begin{pmatrix}0 & 1\\ p\end{pmatrix}$, (viii) $(12u)(23u)(30u)\begin{pmatrix}0 & 4 & 1\\ p & q\end{pmatrix}$,

(ix) $(12u)(23u)(30u)\begin{pmatrix}0 & 1\\ p\end{pmatrix}$,

p, q, r, s being of type i.

19. (i) is reducible since the form of Grade 8 (§ 13),

$$\begin{pmatrix}0 & 2 & 3 & 4 & u\\ x & p & q & r & s\end{pmatrix} \equiv 0,$$

and we may write 1 for u, and $(1u)$ for x in this.

(ii) leads to irreducibles for five conics, if 4 occurs among p, q, r. For four conics it is reducible. In fact, q is 0 or 1, 0 say, and we have the two kinds

$$(01u)\begin{pmatrix}0 & 2 & 3 & 1\\ p & 0 & 2\end{pmatrix}, \quad p = 1 \text{ or } 3.$$

But by writing $(1u)$ for x and 1 for u in the reducible form

$$\begin{pmatrix}0 & 2 & 3 & u\\ x & p & 0 & 2\end{pmatrix}, \quad (p = 1 \text{ or } 3),$$

we reduce this.

(iii) leads to twenty-four irreducible forms for four conics

$$(01u)\begin{pmatrix}0 & 2 & 1\\ 3 & 0\end{pmatrix}. \qquad (a)$$

There are twenty-four such forms, two for each pair 3, 0 of lower links.

For five conics the new type $(01u)\begin{pmatrix}0 & 2 & 1\\ 3 & 4\end{pmatrix}$ comes.

(iv) belongs to less than four conics.

(v) is reducible by writing $(2u)$ for x in $\begin{pmatrix} 0 & 3 & 4 & 1 \\ x & p & q & r & x \end{pmatrix}$, if $q = 0$, $r = 3$ or $q = 1$, $p = 4$.

In other cases it is probably irreducible.

(vi) For four conics there are two types

$$(a) \quad (02u)(21u)\begin{pmatrix} 0 & 3 & 1 \\ 1 & 0 \end{pmatrix},$$

$$(b) \quad (02u)(21u)\begin{pmatrix} 0 & 3 & 1 \\ 2 & 0 \end{pmatrix}.$$

But $\begin{pmatrix} 0 & 3 & 1 & 0 \\ 1 & 0 & p \end{pmatrix}$ is $\equiv 0$, by (A), (§ 7).

Write $(2u)$ for p, and we reduce (a).

Since $\begin{pmatrix} 0 & 3 & 1 \\ x & 2 & 0 & x \end{pmatrix} + \begin{pmatrix} 0 & 1 & 3 \\ x & 2 & 0 & x \end{pmatrix} \equiv 0$, it follows that there are twelve forms (b) which are independent and irreducible, by writing $(2u)$ for x. (β)

For five conics, we have

$$(02u)(21u)\begin{pmatrix} 0 & 3 & 1 \\ 4 & p \end{pmatrix}, \quad p = 0 \text{ or } 2.$$

(vii) We have $(01u)\,2_p - (012)\,u_p = (02u)\,1_p + (21u)0_p$.

Square this. Thus

$$(02u)(12u)(0_3 1) \equiv (012)(01u)(2_3 u).$$

We can take the three forms

$$(012)(01u)(2_3 u), \quad (012)(12u)(0_3 u), \quad (012)(20u)(1_3 u),$$

whose sum is reducible. This gives two irreducible forms for one suffix 3. Thus there are eight altogether. (γ)

(viii) Gives $p = 2, 3, 1$, $q = 2, 3, 0$.

(ix) For four conics $p = 2, 3$; 3 say. By (g_1, g_3) we interchange 0, 1. By $(g_2, 0_3)$, $(g_3, 1_3)$ followed by (g_2, u_3) we interchange 0, 2. Thus the form is symmetrical in 0, 1, 2, and for each suffix 3 we take one form. This gives four terms to the irreducible system. (δ)

For five conics p may be 4.

20. (B) If $[0, 1]$ is a pair of tags. The possible forms are

(i) $(01u)\begin{pmatrix} 0 & 2 & 3 & 4 & u \\ p & q & r & s & \end{pmatrix}(1)$, ($\alpha$) $(12u)(20u)(0)(1)$,

(ii) $(01u)\begin{pmatrix} 0 & 2 & 3 & u \\ p & q & r & \end{pmatrix}(1)$, ($\beta$) $(12u)(23u)(30u)(0)(1)$,

(iii) $(01u)\begin{pmatrix} 0 & 2 & 3 & 4 \\ p & q & r & x \end{pmatrix}(1)$,

(iv) $(01u)\begin{pmatrix} 0 & 2 & 3 \\ p & q & x \end{pmatrix}(1)$,

(v) $(01u)\begin{pmatrix} 0 & 2 & u \\ p & q & \end{pmatrix}(1)$,

(vi) $(01u)\begin{pmatrix} 0 & 2 \\ p & x \end{pmatrix}(1)$, ...

(i) is reducible by writing $(1u)(1)$ for x in

$$\begin{pmatrix} 0 & 2 & 3 & 4 & u \\ x & p & q & r & s \end{pmatrix} \equiv 0.$$

(ii) By $(g, 2_q)$, (ii) $\equiv (21u)\begin{pmatrix} 2 & 0 & 3 & u \\ p & q & r & \end{pmatrix}(1)$,

By $(g, 3_r)$ on (ii), $\equiv (31u)\begin{pmatrix} 3 & 2 & 0 & u \\ q & p & r & \end{pmatrix}(1)$.

Similarly $\equiv (21u)\begin{pmatrix} 2 & 3 & 0 & u \\ q & p & r & \end{pmatrix}(1)$.

Hence $p, q, r \neq 0, 2, 3$, so that they can only be 1 or 4. This is impossible.

Hence also (iii) is reducible and (iv) is $(01u)\begin{pmatrix} 0 & 2 & 3 \\ 1 & 4 & x \end{pmatrix}(1)$.

Thus (iv) gives

$$(01u)\begin{pmatrix} 0 & 2 & 3 \\ 1 & 4 & x \end{pmatrix}\left\{\begin{pmatrix} 1 & 4 & u \\ p & q & \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 4 \\ p & x \end{pmatrix} \text{ or } \begin{pmatrix} 1 & u \\ p & \end{pmatrix} \text{ or } 1_x\right\}.$$

(v) $(01u)\begin{pmatrix} 0 & 2 & u \\ p & q & \end{pmatrix}\begin{pmatrix} 1 & 3 & u \\ r & s & \end{pmatrix}$,

$(01u)\begin{pmatrix} 0 & 2 & u \\ p & q & \end{pmatrix}\begin{pmatrix} 1 & 3 \\ r & x \end{pmatrix}$,

$(01u)\begin{pmatrix} 0 & 2 & u \\ p & q & \end{pmatrix}\begin{pmatrix} 1 & u \\ r & \end{pmatrix}$, ...

For five conics p, q, r, s are 0, 1, ..., 4 in some order.

Consider (v) for four conics. We may take p, q as 3, 1 and $r = 2$.

Let $P = (01u)\begin{pmatrix}0 & 2 & u\\ 3 & 1 & \end{pmatrix}\begin{pmatrix}1 & \dots\\ 2 & \end{pmatrix} \equiv (21u)\begin{pmatrix}2 & 0 & u\\ 3 & 1 & \end{pmatrix}\begin{pmatrix}1 & \dots\\ 2 & \end{pmatrix}$.

By $(g, 0_3)$,

$$P \equiv (20u)\begin{pmatrix}2 & 1 & \dots\\ 3 & 2 & \end{pmatrix}\begin{pmatrix}0 & u\\ 1 & \end{pmatrix} + (210)\begin{pmatrix}2 & u\\ 3 & \end{pmatrix}\begin{pmatrix}0 & u\\ 1 & \end{pmatrix}\begin{pmatrix}1 & r\dots\\ 2 & \end{pmatrix}.$$

The first of these terms is reducible by $(g, 1_2)$, the second is reducible by (g, r_2), whether $r = u$ or 3. For, if $r = u$, it gives

$$P \equiv (21u)\begin{pmatrix}2 & u\\ 3 & \end{pmatrix}\begin{pmatrix}1 & 0 & u\\ 2 & 1 & \end{pmatrix} \equiv 0 \quad [\text{by } (g, 0_1)].$$

If $r = 3$, it gives $P \equiv (213)\begin{pmatrix}2 & u\\ 3 & \end{pmatrix}\begin{pmatrix}1 & 0 & u\\ 2 & 1 & \end{pmatrix}(3)$

$$\equiv 0 \quad [\text{by } (g, 0_1).$$

Thus $(01u)\begin{pmatrix}0 & 2 & u\\ 3 & 1 & \end{pmatrix} 1_x$ is left. By $(g, 2_1)$ 2 and 0 are interchangeable.

By means of $(01u)(\bar{3}x\bar{1})$ we express this last form as

$$(01u)2_3.2_1\{(01u)(\bar{3}x\bar{1}) + 0_3.1_1.u_x + \dots\}$$
$$\equiv (01u)2_3.2_1\{0_1.u_3.1_x - 0_x.u_1.1_3\}$$
$$\equiv (01u)\begin{pmatrix}0 & 2 & u\\ 1 & 3 & \end{pmatrix} 1_x - (01u)\begin{pmatrix}1 & 2 & u\\ 3 & 1 & \end{pmatrix} 0_x.$$

The latter is reducible by $(g, 2_1)$. Hence

$$(01u)\begin{pmatrix}0 & 2 & u\\ 3 & 1 & \end{pmatrix} 1_x \equiv (01u)\begin{pmatrix}0 & 2 & u\\ 1 & 3 & \end{pmatrix} 1_x.$$

This leads to twelve forms, two for each pair of suffixes 1, 3. These are all that occur for four conics under type (γ). (ϵ)

21. (vi) for four conics. The tag (1) is not longer than (0).

The types are
- (a) $(01u)(0_3 2_x)(1_0 3_x)$,
- (b) $(01u)(0_p 2_x)(1_q u)$,
- (c) $(01u)(0_p 2_x)1_x$,
- (d) $(01u)(0_p u)(1_q u)$,

and forms for less than four conics.

(a) By $(g, 3_0)$, $\equiv (013)(1_0 u)3_x(0_3 2_x)$.

By $(g, 2_3)$, $\equiv (023)(0_3 1_0 u)2_x 3_x$.

By $(g, 1_0)$, $\equiv 0$.

(a) is the general type, as 0, 2 and 1, 3 are both interchangeable in the upper links.

(b) 0, 2 are again interchangeable. Two types according as p or q is 3.

Take $p = 1$, $q = 3$. By $(g, 2_1)$, this

$$\equiv (012)(0_1 u)(1_3 u)2_x.$$

By (g, u_3), $$\equiv (01u)(0_1 u)(1_3 2_x)+(12u)\,2_x\,(1_3 0_1 u)\,;$$

the latter is reducible by $(g, 0_1)$. Thus

$$(01u)(0_1 2_x)(1_3 u) \equiv (01u)(1_3 2_x)(0_1 u),$$

so the types are equivalent.

From the identity,

$$(01u)\,2_1 u_3+(u12)\,u_3 0_1 \equiv (01u)\,u_1 2_3+(0u2)\,1_3 u_1+(u12)\,0_3 u_1,$$

we deduce $(u12)(2_1 0_3 u)1_x \equiv (01u)(0_3 2_1 u)1_x+(0u2)(0_3 1_x)(2_1 u)$

$$\equiv (21u)(2_3 0_1 u)1_x+ \quad ,, \quad ,,$$

[this is of type (b)].

Thus the form (b) is reducible to (ϵ) of previous page.

(c) Here $p = 3$, and 0, 2 are interchangeable, giving twelve forms. (ζ)

(d) leads to twelve forms $(01u)(0_2 u)(1_3 u)$. (η)

The forms (a)–(η) are those for four conics with no full brackets.

22. (a) $(12u)(02u)(0)(1)$.

For four or five conics, by $[g_2, (1)] \equiv (12u)(01u)(0)(2)$. By a series of such identities we see that it is symmetrical in 0, 1, 2. Hence (0) is $(0_3 \ldots)$ or $(0_4 \ldots)$; and (1) must be similar. We have

$$(12u)(02u)(0_3 \ldots)(1_4 \ldots),$$

$$(12u)(02u)(0_3 \ldots)\,1_x,$$

$$(12u)(02u)\,0_x 1_x,$$

or $$(12u)(02u)\{(0_3 4_x) \text{ or } (0_3 u)\}\,\{(1_4 3_x) \text{ or } (1_4 u)\},$$

$$(12u)(02u)(0_3 4_x)1_x,$$

$$(12u)(02u)(0_3 u)1_x.$$

For four conics the only one left is $(02u)(12u)(0_3 u)1_x$, which leads to four forms of the system. (θ)

(β) $(12u)(23u)(30u)(0)(1)$. As above, 0, 1, 2, 3 are interchangeable. The only cases are

$$(12u)(23u)(30u)\,0_x 1_x,$$

$$(12u)(23u)(30u)(0_4 u)\,1_x.$$

But $$(23u)\,1_x + (31u)\,2_x + (12u)\,3_x \equiv 0,$$

$$(31u)\,0_x + (10u)\,3_x + (03u)\,1_x \equiv 0.$$

Transfer the terms involving 3_x to the right side; multiply.

Hence $$0_x 1_x\,(23u)(31u) + (31u)(03u)\,1_x 2_x \equiv 0,$$

or $$(02u)(23u)(31u)\,0_x 1_x + (02u)(31u)(03u)\,1_x 2_x \equiv 0,$$

or $$\text{,,} \qquad + (02u)(23u)(31u)\,0_x 1_x \equiv 0 \text{ [by } (g_3,\ 2_x)].$$

Thus the term is reducible.

II. *Forms with one Full Bracket.* (§§ 23–29.)

23. Let $P \equiv (012)M$ be such a form. Suppose that (ij) stand for either (iju) or $(i_k j)$, where $k = i$ or x. Then P is made up of (012), (ij), and i'_x. For, if u occur in a tag, as $(i \ldots j_n u)$, we may reduce P to the preceding case by using $\Sigma(012)(u_{j_n})$.

It is evident that every case is typified by

$$\text{(A)} \quad (012)(01)M,$$

$$\text{(B)} \quad (012)(03)M',$$

where in M' no factor (01), (21), (20) comes.

Neglecting terms which are obviously reducible, the possibilities are as follows :—

(A) $(012)(01)\,2_x$

,, ,, $(20)\,0_x$

,, ,, $(20)(03)\,3_x$ ..... Reducible by (g_1, 3 of g_4)

,, ,, $(20)(03)(34)\,4_x \ldots$,, ,,

,, ,, $(23)\,3_x$

,, ,, $(23)(30)\,0_x$ ..... [,, (g_3, 0_x) to third form above]

,, ,, $(23)(30)(04)\,4_x \ldots$ [,, (g_1, 4 of g_5)]

,, ,, $(23)(34)\,4_x$

,, ,, $(23)(34)(40)\,0_x \ldots$ [,, (g_4, 0_x) to the last but two form],

(B) $(012)(03)(13)2_x$

„ „ $(13)(23)3_x$..... Reducible to type (A) by $(g_1, 3_x)$

„ „ $(13)(23)(34)4_x$ ... „ by $(g_1, 4 \text{ of } g_5)$

„ „ „ $(24)4_x$

„ „ „ $(24)(43)3_x$ ... „ to type (A) by $(g_1, 3_x)$

„ „ „ „ $(40)0_x$ ... [„ „ „ $(g_4, 0_x)$]

„ „ „ „ $(41)1_x$ ... [„ „ „ $(g_4, 1_x)$]

„ „ $(14)(43)(42)4_x$ ... „ „ „ $(g_1, 4_x)$

„ „ $(14)(23)4_x$

„ „ „ $(34)2_x$

„ „ „ $(34)(23)3_x$ ... „ „ „ $(g_1, 3_x)$.

The bracketed remarks only applying if the term g_i involved in the reduction is (pqu) in form. In these forms (A), (01) may be $(01u)$ or $(0_p 1)$.

First we work out all the possibilities for four conics.

One Full Bracket for Four Conics.

24. The possibilities are

$(012)(01)2_x$,

$(012)(01)(20)0_x$,

$(012)(01)(23)3_x$,

$(012)(01)(2_j 3)(30)0_x$,

$(012)(03)(13)2_x$.

Taking these in order, we have

(i) $(012)(0_3 1)2_x$, giving 12 independent irreducibles. (α)

(ii) $(012)(01u)(2_3 0)0_x$; here $(g_2, 2_3)$ applies and shews that there are 12 such. (β)

(iii) $(012)(0_2 1_3 2)1_x$, reducible by using $(012)(\bar{2}x\bar{3})$ as in § 6.

(iv) $(012)(01u)(23u)3_x$ and $(012)(03u)(31u)2_x$; equivalent by $(g_2, 3_x)$.

(v) $(012)(01u)(2_j 3_x)$.

(vi) $(012)(0_p 1)(23u)3_x$.

(vii) $(012)(03u)(1_p 3)2_x$.

(viii) $(012)(0_p 1)(2_q 3_x)$, by $(g_1, 3_q) \equiv (032)(0_p 1_0 2) 3_x$, taking $q = 0$. By using $(032)(\bar{p}x\bar{0})$, we express

$$(032)(0_2 1_0 2) 3_x \equiv (032)(0_2 1_0 3) 2_x + (032)(2_0 1_2 3) 0_x.$$

Thus we may take $(012)(0_1 3_0 2) 1_x$ as a typical irreducible form, in which 2, 3 are interchangeable by $(g, 3_1)$. So there are twelve of these. (γ)

(ix) $(012)(01u)(2_j 3_k 0_x)$, by $(g_1, 3_k)$ $k \neq 2$. By $(g_2, 3_k)$ $k \neq 1$, so form is reducible.

(x) $(012)(01u)(30u)(2_j 3) 0_x$, by $(g_2, 2_j)$ $j \neq 1$, so take $j = 0$. The form

$$\equiv (012)(01u)(302)(u_0 3) 0_x.$$

But $(031) 2_0 \equiv (021) 3_0 + \ldots$ and $(031)(20u) \equiv (021)(30u) + \ldots$.

Hence, by multiplication,

$$0 \equiv (012)(032)(03u)(1_0 u) 0_x + (012)(302)(01u)(3_0 u) 0_x.$$

Of these the first $\equiv$ second term, by $(g_3, 1_0)$, so it is reducible.

(xi) $(012)(1_p 0_x)(2_j 3)(30u)$, $(g_4, 1_p)$ shews $p \neq 3$; if $p = 2$, $(g_4, 1_p)$, followed by $(g_1, 3_p)$ reduces it.

(xii) $(012)(1_p 0_k 3_j 2) 0_x$, if $k = 2$, $(g, 3_k)$ reduces it: if $k = 1$, then $j = 0$, and it is reducible, since

$$\begin{pmatrix} 1 & 0 & 3 & u \\ x & p & 1 & 0 \end{pmatrix} \equiv 0, \quad \S\ 7\ (\text{A}).$$

(xiii) $(012)(0_p 3_q 1) 2_x$, same as (viii). But p or q might be x (the only case in the above list in which the form is not obviously reducible). We obtain

$$(012)(0_1 3_x) 1_x 2_x \equiv (013)(0_1 2_x) 1_x 3_x \equiv (213)(2_1 0_x) 1\ 3_x \ldots,$$

or four forms, one for each suffix. (δ)

We have still to consider forms (iv), (v), (vi), (vii).

(iv) $(012)(03u)(31u) 2_x$ or (0123) say. There are 12 of these, since

$$(0123) = -(1023).$$

They satisfy

$$(0123)-(0132) = (1203)-(1230) = (2013)-(2031)$$
$$= (2301)-(2310) = (3021)-(3012)$$
$$= (1320)-(1302) = \lambda \text{ say.}$$

For $(0123)-(1203) = (012)(02u)(31u)3_x$

$$= (0132)-(1230), \text{ by } (g_1, 3_x).$$

So seven forms should be taken for the irreducible system. (ϵ)

(v), (vi), (vii) Take x as suffix. We then have $(012)(03u)1_x2_x3_x$ as type. Call this r_{03} and $(iju)(ikl)$ ..., r_{ij}, where $ijkl$ is an even number of inversions of 0123.

By $(g_1, 2_x)$, $r_{01} \equiv r_{21}$, similarly $\equiv r_{31}$.

By $(g_2, 1_x)$, $r_{01} \equiv r_{10}$. Hence

$$r_{01} \equiv r_{21} \equiv r_{12} \equiv r_{02} \quad \text{and} \quad r_{01} \equiv r_{03},$$

So $3r_{01} = r_{01}+r_{02}+r_{03} = 0$ by the identity, so $r_{ij} \equiv 0$.

The other cases are

$(012)(01u)(2_0 3_x)$, (a) } By $(g_1, 3_0)$, (c) is expressible in terms of type (a).
$(012)(03u)(2_0 1)3_x$, (b) } By $(g_1, 3$ of $g_2)$, (b) ,, ,, ,, (c).
$(012)(03u)(1_0 3)2_x$, (c) }

So all are included in (a). By $(g_2, 2_0)$ 1, 2 are interchangeable in (a). Thus there are twelve forms (a), three for each value of suffix. (ζ)

Hence forms involving one full bracket for four conics, which have not been reduced, are (α), (β), (γ), (δ), (ϵ), (ζ).

We now find the possible types for five conics.

25. We deal with the above list of § 23 according as g_2 is of type (iju) or (i_kj).

(A) First consider $(012)(01u)M$.

We have

$(012)(01u)(2_4 3_x)$

,, ,, $(2_j 3_k 0_x)$

,, ,, $(2_j 3)(30u)0_x$

,, ,, $(2_j 3_k 4_x)$

,, ,, $(23u)(3_k 4_x)$

,, ,, $(2_j 3)(34u)4_x$

,, ,, $(23u)(3_j 4)(40u)0_x$ [R by $(g_3, 0_x)$ followed by $(g_1$, 4 of $g_5)$]

,, ,, $(2_i 3_j 4)(40)0_x$

,, ,, $(23u)(3_j 4_k 0_x)$ [R by $(g_3, 0_x)$,, $(g_1$, 4 of $g_5)$].

In $(012)(01u)(2_i 3_j 4)(40)0_x$, $i, j \neq x$, also $(g_2$, 2_i or $3_j)$ shew that $i, j \neq 1$. By [g_2, 4 of (40)] followed by $(g_1, 3_j)$, we see that $j \neq 2$; therefore $j = 0$ and $i = 4$, and this is reduced by $(g_1, 3_i)$.

26. (A′) Next, consider $(012)(0_p 1)M$, where $(01u)$, $(02u)$, or $(12u)$ are not in M. We have

(*a*) $(012)(1_p 0_q 2)0_x$,

(*b*) ,, $(0_p 1)(23u)3_x$,

(*c*) ,, ,, $(2_q 3_x)$,

(*d*) ,, ,, $(23u)(3_q 0)0_x$,

(*e*) ,, $(1_p 0_q 3_r 2)0_x$,

(*f*) ,, $(0_p 1)(23u)(3_r 4_x)$,

(*g*) ,, ,, $(23u)(34u)4_x$,

(*h*) ,, ,, $(2_q 3)(34u)4_x$,

(*i*) ,, ,, $(2_q 3_r 4_x)$,

$(012)(0_p 1)(23u)(3_r 4)(40u)0_x$ } reducible by $(g_3, 0_x)$ to $(012)(20u)M$,
,, ,, $(4_s 0_x)$ } *i.e.*, to last set,
$(2_q 3_r 4)(40u)0_x$,
$(2_q 3_r 4_s 0_x)$.

27. These last two forms are reducible.

Take $(012)(2_q 3_r 4_s 0_p 1)0_x$. Here $r \neq 2$, $s \neq 2$, $s \neq 3$, by $(g_1, 4_s)$, and similar reductions. Hence $s = 1$, $r = 0$, and the form is of the reducible type

$$\begin{pmatrix} 1 & 0 & p' & z \\ y & q' & 1 & 0 \end{pmatrix}.$$

The last but one is

$(012)(0_p 1)(2_q 3_r 4)(40u)0_x$ [by $(g_1, 3_r)$, $r \neq 2$, also $r \neq x$]

$\equiv (013)(0_p 1)(3_q 2_r 4)(40u)0_x$

$\equiv (014)(0_p 1)(3_q 2_r 4)(30u)0_x$

$\equiv (012)(0_p 1)(3_q 4_r 2)(30u)0_x$ [hence $r = 0$ or 1, $q = 0$ or 1, let $q = 0$, $r = 1$]

$\equiv (032)(0_p 1)(1_0 4_1 2)(30u)0_x$

$\equiv (034)(0_p 1)(1_0 2_1 4)(30u)0_x$

$\equiv (031)(0_p 4)(1_0 2_1 4)(30u)0_x$

$\equiv 0$ by $(g_1, 2_1)$.

Thus the forms (a)–(i) are left.

28. Consider the type (B). First $(012)(03u)M$. We have

$(012)(03u)(1_j 3)2_x$ $(j = 4)$

,, ,, $(13u)(24u)4_x$

,, ,, $(13u)(2_j 4_r)$

,, ,, $(1_i 3)(2_j 4_x)$

,, ,, $(1_i 3)(24u)4_x$

,, ,, $(13u)(2_j 4)(40u)0_x$

,, ,, $(13u)(2_j 4_k 0_x)$ } reducible by $(g_3, 0_x)$

,, ,, $(1_i 3)(2_j 4)(40u)0_x$ [reducible to type (A) by $(g_5, 1_i)$]

,, ,, $(1_i 3)(2_j 4_k 0_x)$ [,, ,, $(g_2, 4_k)$, followed by $(g_1, 3_k)$],

four more with 1_x for 0_x at the end, reducible by $(g_2, 1_x)$.

$$(012)(03u)(14u)(23u)4_x$$
$$\text{,,} \quad \text{,,} \quad (1_p4)(23u)4_x$$
$$\text{,,} \quad \text{,,} \quad (1_p4)(2_q3)4_x$$
$$\text{,,} \quad \text{,,} \quad (14u)(2_q3)4_x$$
$$\text{,,} \quad \text{,,} \quad (14u)(34u)2_x$$
$$\text{,,} \quad \text{,,} \quad (1_p4)(34u)2_x$$
$$\text{,,} \quad \text{,,} \quad (1_p4_q3)2_r$$
$$\text{,,} \quad \text{,,} \quad (14u)(3_q4)2_x.$$

(B) Consider the type $(012)(0_p3)M$.

29. Neglecting types already written down in the last set, we have

$$(012)(0_p3_q1)M, \qquad (012)(0_p3)(1_q4)M,$$

where no factor $(03u)$, $(13u)$, $(23u)$, $(04u)$, ... comes, or

(*a*) $(012)(0_p3_q1)2_x$,

(*b*) $(012)(0_p3_q1)(2_r4_x)$,

(*c*) $(012)(0_p3_q1)(2_r4_s0_x) = (012)(2_r4_s0_p3_q1)0_x$, $r \ldots q \neq x$,

(*d*) $(012)(0_p3_q1_r4_s2)1_x$,

(*e*) $(012)(0_p3)(1_r4_x)(3_q2)$, same as (*b*) in type,

(*f*) $(012)(0_p3_q4_r1)2_x$.

But $\begin{pmatrix} i & j & k & l & u \\ x & p & q & r & s \end{pmatrix}$ is reducible for five conics (§ 13); hence

$$\begin{pmatrix} & 2 & 4 & 0 & 3 & 1 \\ (01) & r & s & p & q & \end{pmatrix} 0_x \equiv 0,$$

where (01) is put for x, and 1, 0_x for u. This reduces (*c*). (*d*) is the same type.

III. *Forms with two Full Brackets.* (§§ 30–34).

30. We take first of all the type

$$P = (012)(013)M \quad (\S\ 16).$$

The possibilities are

$$(012)(013)\begin{cases}(23)\\ (20)(01)(13) \ \ldots\ldots\ R \text{ by } (g_1, 3 \text{ of } g_5)\\ (20)(01)(14)(43)\ldots\ R \text{ by } (g_1, 3 \text{ of } g_6)\\ (20)(03)\\ (20)(04)(41)(13)\ldots\ R \text{ by } (g_1, 3 \text{ of } g_6)\\ (20)(04)(43)\\ (24)(40)(01)(13)\ldots\ R \text{ by } (g_1, 3 \text{ of } g_6)\\ (24)(40)(03)\ \text{(same type as last but one)}\\ (43)\end{cases}$$

Thus

$$\left.\begin{array}{l}(012)(013)(23)\\ \text{,,}\quad\text{,,}\quad(20)(03)\\ \text{,,}\quad\text{,,}\quad(20)(04)(43)\\ \text{,,}\quad\text{,,}\quad(24)(43)\end{array}\right\}\ \text{are left.}$$

That is to say, we have

$$\left.\begin{array}{ll}(a) & (012)(013)(2_p3)\\ (b) & (2_p0_q3)\\ (c) & (2_p0_q4_r3)\\ (d) & (2_p4_q3)\end{array}\right\}\ p, q, r \text{ being } i \text{ or } x,$$

$$\begin{array}{ll}(a_1) & (012)(013)(23u)\\ (a_2) & (20u)(0_p3)\\ (a_3) & \text{,,}\quad(0_p4_q3)\\ (a_4) & (24u)(4_p0_q3)\\ (a_5) & (24u)(4_p3)\\ (a_6) & (04u)(43u)(2_p0)\\ (a_7) & (20u)(04u)(4_p3)\\ (a_8) & (20u)(43u)(0_p4)\end{array}$$

31. First take the case of four conics.

The only types are $(012)(013)(23)$

$(012)(013)(20)(03)$.

Thus we have

(i) $(012)(013)(23u)$. If $q_{ij} = (ijk)(ijl)(klu) = -q_{ji}$ and $ijkl$ = even number of inversions of 0123, then

$$q_{12}-q_{02}+q_{01} = 0, \quad q_{23}-q_{03}+q_{02} = 0, \quad q_{31}-q_{01}+q_{03} = 0.$$

So all are expressible in terms of three of them, q_{01}, q_{02}, q_{13}. (α)

(ii) $(012)(013)(2_x 3) = p_{01} = p_{10}$. Then $p_{01}+p_{02}+p_{12} = 0$, so p_{12}, p_{23}, p_{31} are expressible in terms of p_{01}, p_{02}, p_{03}, and the sum of the latter is zero. We take two of the set for our system. (β)

(iii) $(012)(013)(2_0 3)$ is reducible by squaring

$$(012)3_0-(013)2_0 \equiv (032)1_0.$$

(iv) $(012)(013)(20u)(30u)$ is similarly reducible.

(v) $(012)(013)(20u)(3_p 0)$, $(g_1, 3_p)$ reduces this if $p = 1$, $(g_2$, 2 of $g_3)$ shews that 1 and 2 are interchangeable, so it is reducible if $p = 2$. So let $p = x$. Since

$$(012)3_x+(023)1_x = (123)0_x-(031)2_x,$$

and $$(021)(03u)0_x+(032)(01u)0_x = -(013)(02u)0_x,$$

therefore $$(012)(032)(01u)0_x 3_x+(021)(023)(03u)1_x 0_x \equiv 0,$$

or $$(012)(031)(02u)0_x 3_x+(021)(013)(03u)2_x 0_x \equiv 0,$$

or $$\text{,,} \qquad +(021)(013)(02u)3_x 0_x \equiv 0,$$

so $$(012)(013)(20u)(3_x 0) \equiv 0.$$

(vi) $(012)(013)(2_p 0_q 3)$, by $(g_1, 3_q)$ and $(g_2, 2_p)$, p, $q \neq 1$, so $p = 3$, $q = 2$ or x. But

$$(012)3_p-(013)2_p = (032)1_p+(312)0_p,$$

$$(013)2_3-(012)3_3 = (023)1_3+(213)0_3.$$

Multiply these.

If $p = 2$, we reduce (vi) ; if $p = x$, we reduce (vi).

Thus (α), (β) are the only types for four conics.

32. Detailed reduction of the above sets, § 30, for five conics.

(γ) Take $(012)(013)(2_p 0_q 4_r 3)$,

$$p = x, 1, 3, 4, \qquad q = x, 1, 2, 3, \qquad r = x, 0, 1, 2.$$

By $(g_1, 3_r)$, this $\equiv (032)(013)(2_p 0_q 4_r 1)$. Hence $r \neq 1$. Similarly $p \neq 3$.
By $(g_1, 4_q)$, $q \neq 1$ or 3. Hence $q = x$ or 2.
The form $\equiv (012)(023)(1_p 0_q 4_r 3)$ by $(g_2, 2_p)$ (hence $p \neq 1$).
Hence $q \neq 2$, by $(g_1, 4_q)$. Thus $q = x$, $p = 4$, $r = 0$ or 2. But, by $(g_1, 3_r)$ on last written form, $r \neq 2$; hence $r = 0$. The form is now

$$(012)(013)(2_4 0_x 4_0 3) \equiv (012)(023)(1_4 0_x 4_0 3)$$
$$\equiv (013)(023)(1_4 0_x 4_0 2) \equiv (014)(023)(1_4 0_x 3)(4_0 2),$$

and, by (g_1, g_2),

$$\equiv (012)(043)(1_4 0_x 3)(4_0 2) \equiv 0, \text{ by } (g_2, 1_4).$$

Thus it is reducible.

(d) $(012)(013)(2_p 4_q 3)$, $\quad p = x, 1, 3, 0,$
$\quad q = x, 1, 2, 0.$

(b) $(012)(013)(2_p 0_q 3)$. Here $p \neq 1$, $q \neq 1$,
$\quad p = 4, x, 3, \quad q = 4, x, 2.$

(a_2) $(012)(013)(20u)(0_4 3)$.

(a_3) If p, q contain 1, 2, this is reducible as (v) of § 30 (a). Hence we have
$(012)(013)(20u)(0_3 4_0 3)$ and $(012)(013)(20u)(0_3 4_x 3)$.

The former $\equiv (032)(013)(20u)(0_3 4_0 1) \equiv 0$, by $(g_1, 4_3)$.

(a_4) $(012)(013)(24u)(4_p 0_q 3)$ (here $p \neq 1$, $p = x, 2, 3$, $q = x, 4, 2, 1$)
$\equiv (032)(013)(24u)(4_p 0_q 1)$ (hence $p \neq 3$, $q \neq 1$)
$\equiv (032)(014)(24u)(3_p 0_q 1)$ (hence $q \neq 2$).

Either $p = 2$, $q = 4$ or p, q contain x. In the first case,

$$(012)(013)(24u)(4_2 0_4 3)$$
$$\equiv (012)(043)(21u)(4_2 0_4 3) + (012)(023)(14u)(4_2 0_4 3)$$
$$\equiv (042)(043)(21u)(1_2 0_4 3) + \text{a reducible term, by } (g_2, 4_2)$$
$$\equiv 0, \text{ by } (g_1, 3_4).$$

Hence p or $q = x$.

(a_5) $(012)(013)(24u)(4_p 3)$ $\quad (p = x, 0, 1, 2).$

(a_6) $(04u)(43u)(2_p 0)$ $\quad (p = x, 3, 4).$

(a_7) $p = x, 2, 0.$

(a_8) $p = x, 1, 2, 3.$

33. There is the second type for five conics (§ 16)

$$P = (012)(034)M.$$

Here we may have

	Reducible to the previous form by
(012)(034)(12)(34)	
,, ,, ,, (30)(04)	(g_1, 4 of g_5)
,, ,, ,, ,, (01)(14)	P
,, ,, ,, ,, (02)(24)	P
,, ,, ,, (31)(14)	
,, ,, ,, ,, (10)(04)	P
,, ,, (13)(24)	
,, ,, ,, (20)(04)	(g_1, 4 of g_5)
,, ,, ,, ,, (01)(14)	P
,, ,, ,, ,, (03)(34)	P

where P stands for the perpetuant reduction of

$$(012)(034)1_p 2_q 3_r 4_s \quad (\S\ 34). \tag{1}$$

Thus (012)(034)(12)(34), (13)(24), (12)(31)(14) } are left.

We may have (i) $g_1 g_2 (12u)(34u)$,

(ii) ,, $(12u)(3_p 4)$ $(p = i, x)$;

but not $g_1 g_2 (1_p 2)(3_q 4)$ on account of the reduction (1).

We also have (iii) $g_1 g_2 (31u)(24u)$,

(iv) $(31u)(2_p 4)$,

(v) $(12u)(31u)(14u)$,

(vi) $(12u)(31u)(1_p 4)$,

(vii) $(12u)(3_p 1)(1_q 4)$,

(viii) $(31u)(14u)(1_p 2)$,

(ix) $(31u)(1_p 4)(1_q 2)$.

(vii) and (ix) are reducible. For, by (g_1, 3_p) on (vii), we shew that, if $p = 0$, the form is of type $(012)(013)M$, which is already considered.

Thus $p, q \neq 0$. Using a succession of such reductions and also (g_1, 2 of g_3), we rule out all cases unless $p = x$. But then (g_2, 1 of g_3) reduces it. Similarly for (ix).

Hence (i)–(vi), (viii) are the system required.

Perpetuant Form.

34. (012)(034) considered as a perpetuant type of degree 5 is reducible to $\Sigma(012)(013)$ of degree 4. The proof requires that the order of the forms should be two at least. In fact,

$$(012)(034)\,1_p 2_q 3_r 4_s = \Sigma \pm (012)(013)\,2_{p'} 3_{q'} 4_{r'} 4_{s'},$$

p', q', r', s' being p, q, r, s in some order.

By the identity (g_1, 3 of g_2),

$$(012)(034) - (032)(014) - (013)(024) \equiv 0. \qquad (1)$$

By (g_1, 3_r), we have

$$(012)(034) - (013)(034) + (023)(034) - (034)(123) = 0,$$

or $$(012)(034) \equiv (123)(034).$$

Hence $$(012)(034) \equiv (012)(134) \equiv (032)(134) \equiv (032)(104).$$

But $$(012)(134) \equiv (042)(134) \equiv (042)(130).$$

Hence $$(012)(034) \equiv (032)(104) \equiv (042)(130).$$

By the use of identity (1), $(012)(034) \equiv 0$,

i.e., is reducible.

IV. *Forms with three Full Brackets.* (§§ 35–38.)

35. We have shewn that these fall into four classes,

(i) $(023)(031)(012)M$,

(ii) $(012)(013)(014)M$,

(iii) $(012)(013)(034)M$,

(iv) $(012)(013)(234)M$.

(i) alone occurs for four conics. Consider this first.

We may have

(*a*) $(023)(031)(012)0_x$,

„ „ „ $(0, i)M$, reducible by $[g_i, i$ of $(0, i)]$, if $i = 1, 2, 3$,

(*b*) $(023)(031)(012)(04u)4_x$,

(*c*) „ „ „ $(0_j 4_x)$ $(j = 1, 2, 3)$.

Forms (*a*), (*b*), (*c*) are the possible types.

Next, let $P = (012)(013)(014)M$.

Here 2 appears in M, but not in the association (2, 3), (2, 4), owing to $g_1 g_2$, $g_1 g_3$ respectively. Thus we may only have (20), (21), 2_x. Similarly for 3, 4.

There is therefore one type

$$(012)(013)(014)(20)(31)4_x.$$

By (g_3, 2 of g_4), followed by (g_2, 2 of g_3), we reduce this. Thus it leads to no irreducible forms.

36. The next type is $P = (012)(013)(024)M$.

There is another 0. If it is linked with 1, (g_3, and this 1) reduces the form to the previous case. Similarly, if it is linked with 2. The cases of 3 and 4 are alike, so we take for consideration

(A) $(012)(013)(024)0_x M$,

(B) $(012)(013)(024)(03)M$.

In (A), (31), (24) cannot appear owing to g_3.

In (B), (1) cannot appear in M owing to g_2.

We have

(A) $(012)(013)(024)0_x(34)$

„ „ „ „ $(30)(04)$,

(B) „ „ „ $(03)4_x$

„ „ „ „ $(42)2_x$ [reducible by (g_2, 2 of g_5)]

„ „ „ „ $(42)(23)3_x$ [„ (g_3, 3 of g_6)]

„ „ „ „ $(43)3_x$ [„ (g_1, 3_x), followed by (g_1, 0 of g_4)]

„ „ „ „ $(43)(32)2_x$ [reducible by (g_3, 3 of g_6)].

As for $(012)(013)(024)(30)(04)0_x$. We have

$$(013)(024)-(012)(034) \equiv (023)(014),$$

and

$$(012)(30) \ -(013)(20) \ \equiv (032)(10);$$

hence $(012)(013)(024)(30)(04)0_x+(012)(013)(034)(20)(04)0_x \equiv 0.$

These terms are equivalent by (g_3, 2 of g_4) acting on the second. Hence they are reducible.

Thus the forms of this type are reducible to preceding forms, except

$(012)(013)(024)(34)0_x$, where (34) is $(34u)$ or (3_i4) or 3_x4_x,

„ „ „ $(03)4_x$, „ (03) is $(03u)$, (0_i3), or 0_x3_x.

37. The last type is $P = (012)(013)(234)M$.

If M contain 0, this form is expressible in previous types by using the identity

$$(234)0_p = (034)2_p+\dots .$$

Thus no 0 occurs in M; similarly no 1, 2, or 3 occurs in M. The possibility is

$$(012)(013)(234)4_x.$$

This form is unreduced even when 0, 1, 4 refer to the same conic, so that the presumption that it is irreducible in any case is strong.

38. Summing up for this case of three full brackets, we have

For four conics, four forms

$$(023)(031)(012)0_x.$$

For five conics,

(i) $(023)(031)(012)(04u)4_x$,

(ii) $(023)(031)(012)(0_j4_x)$,

(iii) $(012)(013)(024)(34u)0_x$,

(iv) $(012)(013)(024)(3_i4)0_x$,

(v) $(012)(013)(024)(03u)4_x$,

(vi) $(012)(013)(024)(0_i3)4_x$,

(vii) $(012)(013)(234)4_x$.

39. *The Complete System for Four Conics.*

FORMS BELONGING TO LESS THAN FOUR CONICS ARE INSERTED FOR COMPLETENESS.

Reference number in previous pages when used in them.	Number of irreducible forms.	Abbreviated symbol of the form		Ordinary symbol.	Number of conics needed.
	4 + 4	0_x^2	*	$a_x^2 = b_x^2 = c_x^2,\ a_x'^2,\ a_x''^2,\ a_x'''^2$	1
	4	0_0^2		$(abc)^2$	
	12	0_1^2		$a_{a'}^2$,	2
	12	$\begin{pmatrix} 0 & u \\ x & 1 \end{pmatrix}$		$a_x a_{x'} u_{a'}$	
	6 + 6	$(01u)^2$	*	$(aa'u)^2$	
(*a*)	12 + 12	$(01u)\begin{pmatrix} 0 & u \\ 1 \end{pmatrix} 1_x$	*	$(aa'u)\, a_{a'} u_{a'} a_x$	
(*b*)	6 + 6	$(01u)\begin{pmatrix} 0 & u \\ 1 \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 \end{pmatrix}$	*	$(aa'u)\, a_{a'} a'_a u_{a'} u_a$	
(*c*)	6 + 6	$(01u)\, 0_x 1_x$	*	$(aa'u)\, a_x a'_x$	
(*d*)	24	$\begin{pmatrix} 0 & 1 & u \\ x & 2 & 0 \end{pmatrix}$		$a_x a_{a''} a'_{a''} a'_a u_a$	3
(*e*)	12 + 12	$\begin{pmatrix} 0 & 2 \\ x & 1 & x \end{pmatrix}$	*	$a_x a_{a'} a''_{a'} a''_x$	
(*f*)	12 + 12	$(01u)\begin{pmatrix} 0 & 1 \\ 2 \end{pmatrix}$	*	$(aa'u)\, a_{a''} a'_{a''}$	
(*g*)	12 + 12	$(01u)\begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 \end{pmatrix}$	*	$(aa'u)\, a_{a'} a''_{a'} a''_a a'_a$	All these forms (*d*) to (*p*) are found by Ciamberlini (*Battaglini*, Vol. XXIV).
(*h*)	24 + 24	$(01u)\begin{pmatrix} 0 & u \\ 2 & 1 \end{pmatrix} 1_x$	*	$(aa'u)\, a_{a''} u_{a''} a'_x$	
(*i*)	12 + 12	$(01u)\begin{pmatrix} 1 & 2 \\ 0 & x \end{pmatrix} 0_x$	*	$(aa'u)\, a'_a a''_a a''_x a_x$	
(*j*)	24 + 24	$(01u)\begin{pmatrix} 0 & u & 1 \\ 2 & 0 \end{pmatrix}$	*	$(aa'u)\, a_{a''} u_{a''} u_a a'_a$	
(*k*)	4 + 4	$(01u)(12u)(20u)$	*	$(aa'u)(a'a''u)(a''au)$	
(*l*)	4 + 4	$(012)^2$	*	$(aa'a'')^2$ (The only invariant of 3 conics)	
(*m*)	4 + 4	$(012)\, 0_x 1_x 2_x$	*	$(aa'a'')\, a_x a'_x a''_x$	
(*n*)	12 + 12	$(012)\begin{pmatrix} 1 & 2 \\ 0 \end{pmatrix} 0_x$	*	$(aa'a'')\, a'_a a''_a a_x$	
(*o*)	8 + 8	$(012)(01u)\, 2_x$	*	$(aa'a'')(aa'u)\, a''_x$	
(*p*)	12 + 12	$(012)(01u)(02u)\, 0_x$	*	$(aa'a'')(aa'u)(ba''u)\, b_x$	
§ 5	12	$\begin{pmatrix} 0 & 1 & 0 \\ 2 & 3 \end{pmatrix}$		$a_{a''} a'_{a''} a'_{a'''} a_{a'''}$	The only new type of invariant
.....	.....	.....		.....	
§ 6	12 + 12	$\begin{pmatrix} 0 & 1 & 2 \\ x & 3 & 0 & x \end{pmatrix}$	*	$a_x a_{a'''} a'_{a'''} a'_a a''_a a''_x$	4
.....	.....	.....		.....	
§ 7	12	$\begin{pmatrix} 0 & 1 & 3 & u \\ x & 3 & 0 & 2 \end{pmatrix}$		$a_x a_{a'''} a'_{a'''} a'_a a'''_a a'''_{a''} u_{a''}$	One for each grouping 013 top 023 bottom
.....	.....	.....		.....	
(1) § 6	24	$\begin{pmatrix} 0 & 2 & u \\ x & 1 & 3 \end{pmatrix}$		$a_x a_{a'} a''_{a'} a'_{a''} u_{a''}$,	
.....	.....	.....		.....	

* Denotes the existence of a correlative form.

Reference number in previous pages when used in them.	Number of irreducible forms.	Abbreviated symbol of the form	Ordinary symbol.	Number of conics needed.
I (α)	24 + 24	$(01u)\begin{pmatrix}0&2&1\\3&0\end{pmatrix}$ *	$(aa'u)\,a_{a'''}a''_{a'''}a''_{a}a'_{a}$	
(β)	12 + 12	$(02u)(03u)\begin{pmatrix}2&1&3\\0&2\end{pmatrix}$ *	$(aa''u)(aa'''u)\,a''_{a}a'_{a}a'_{a''}a'''_{a''}$	
(γ)	8 + 8	$(02u)(12u)\begin{pmatrix}0&1\\3\end{pmatrix}$ *	$(aa''u)(a'a''u)\,a_{a'''}a'_{a'''}$	
(δ)	4 + 4	$(02u)(23u)(31u)\begin{pmatrix}0&1\\2\end{pmatrix}$ *	$(aa''u)(a''a'''u)(a'''a'u)\,a_{a''}a'_{a''}$	
(ϵ)	12 + 12	$(01u)\begin{pmatrix}0&2&u\\3&1\end{pmatrix}1_x$ *	$(aa'u)\,a_{a''}a''_{a'''}a''_{a'}u_{a}a'_{x}$	
.....	.....	.....	.....	
(ζ)	12 + 12	$(01u)\begin{pmatrix}0&2\\3&x\end{pmatrix}1_x$ *	$(aa'u)\,a_{a}a'_{a'''}a''_{x}a'_{x}$	
.....	.....	.....	.....	
(η)	12 + 12	$(01u)\begin{pmatrix}0&u\\2\end{pmatrix}\begin{pmatrix}1&u\\3\end{pmatrix}$ *	$a_{a''}a'_{a'''}u_{a''}u_{a'''}$	
.....	.....	.....	.....	
(θ)	4 + 4	$(01u)(12u)\begin{pmatrix}0&u\\3\end{pmatrix}2_x$ *	$(aa'u)(a'a''u)\,a_{a'''}u_{a'''}a''_{x}$	
.....	.....	.....	.....	
II (α)	12 + 12	$(012)\begin{pmatrix}0&1\\3\end{pmatrix}2_x$ *	$(aa'a'')\,a_{a'''}a'_{a'''}a''_{x}$	
.....	.....	.....	.....	4
(β)	12 + 12	$(012)(01u)\begin{pmatrix}2&0\\3&x\end{pmatrix}$ *	$(aa'a'')\,(aa'u)\,a''_{a'''}b_{a'''}b_{x}$	
.....	.....	.....	.....	
(γ)	12 + 12	$(012)\begin{pmatrix}0&3&1\\2&0\end{pmatrix}2_x$ *	$(aa'a'')\,a_{a''}a'''_{a''}a'''_{a}a'_{a}a''_{x}$	
.....	.....	.....	.....	
(δ)	4 + 4	$(012)\begin{pmatrix}0&3\\2&x\end{pmatrix}1_x2_x$ *	$(aa'a'')\,a_{a''}a'''_{a''}a'''_{x}a''_{x}a'_{x}$	
.....	.....	.....	.....	
(ϵ)	7 + 7	$(012)(30u)(31u)\,2_x$ *	$(aa'a'')(a'''au)(a'''a'u)\,a''_{x}$	
.....	.....	.....	.....	
(ζ)	12 + 12	$(012)(12u)\begin{pmatrix}0&3\\1&x\end{pmatrix}$ *	$(aa'a'')(a'a'''u)\,a_{a'}a'''_{a'}a'''_{x}$	
.....	.....	.....	.....	
III (α)	3 + 3	$(012)(013)(23u)$ *	$(aa'a'')(aa'a''')(a''a'''u)$	
.....	.....	.....	.....	
(β)	2 + 2	$(012)(013)\,2_x3_x$ *	$(aa'a'')(aa'a''')\,a''_{x}a'''_{x}$	
.....	.....	.....	.....	
IV	4 + 4	$(023)(031)(012)\,0_x$	$(aa''a''')(aa'''a')(ba'a'')\,b_x$	
Total ...	784			

New forms:
- 12 Invariants
- 109 Covariants
- 109 Contravariants
- 154 Mixed concomitants.

Old 400

784

* Denotes the existence of a correlative form.

The Question of the Irreducibility of the Forms retained.

40. Suppose a form containing symbols of three conics to be irreducible; then a form of the same "type" belonging to four conics is most probably irreducible.

If a form of four conics is irreducible when two of the conics are made identical, it is irreducible as it stands, for we have taken away one of the reducing processes without introducing another.

Thus $\begin{pmatrix} 0 & 1 & u \\ x & 2 & 3 \end{pmatrix}$ is irreducible, since $\begin{pmatrix} 0 & 1 & u \\ x & 2 & 0 \end{pmatrix}$ is.

Again $\begin{pmatrix} 0 & 1 & 0 \\ & 2 & 3 \end{pmatrix}$ is irreducible, since $\begin{pmatrix} u & 1 & u \\ & 2 & 3 \end{pmatrix}$ has not been reduced.

I (α) is irreducible with (g) under the list for three conics,

(ξ) ,, (i) ,, ,,

(η) ,, (j) ,, ,,

II (α) ,, (n) ,, ,,

(β) ,, (d) ,, ,,

making 0, 1 the same,

(ϵ) ,, (p) under the list for three conics,

III (α) ,, (f) if conics 0 and 1 are the same,

(β) ,, (e) ,, ,,

IV ,, (n) if conics 0 and 3 are the same.

Again, Iγ would be irreducible even if u and 2 were replaced by 2 and 2, so it is probably irreducible as it is.

IIδ is irreducible, for if

$$(012)(0_2 3_x)1_x 2_x = \lambda(012)0_x 1_x 2_x \,.\, 3_2^2 + \mu(013) \ldots$$

is assumed, and if we find λ, μ, ... by making 0, 1 refer to same conic, and so on, we get self-contradictory equations for λ, μ, ν, ρ.

The remaining forms cannot be similarly shewn to be irreducible; at least there is no obvious way of doing so.

41. Thus the complete system for four conics has been tabulated. The size of the actual number of new forms, 384, is about what one would expect, judging from the rate of increase for lower systems.

For one conic the number is 3.

For two conics ,, 14, which with the separate single systems = 20.

For three conics ,, 76, which with the separate single and double systems = 128.

For four conics ,, 384, which with the separate single, double and treble systems = 784.

We have not actually enumerated the forms for five conics. All that has been done is to write down all the possible types. Incidentally every form of total degree 14 or more has been reduced in the previous pages. This shews that no irreducible type of ternary quadratics exists for degree > 13. It may happen that the few which are left in the system whose degree is 13 are reducible.

Such a form is the correlative of $(012)(0_p 3_q 4_r 1)2_x$, p, q, r being three of 0, 1, 2, 3, 4. In any case there would be a great deal of labour involved in reducing such forms, as they have so far been proof against the methods employed above for forms of higher degree.

For binary forms the corresponding limit of the degree is 3, possessed by the linear invariant (12)(23)(31) or $(bc)(ca)(ab)$ in the usual notation.

The corresponding invariant in ternary forms

$$(012)(034)(135)(245)$$

is only of degree 6, so that there would not be much prospect of dealing with the question of ternary quadratic types in the direct way which applies to binary forms (Grace and Young, p. 322).

AN EXPRESSION OF $(1-z)^{-1}$ BY MEANS OF POLYNOMIALS

By H. F. BAKER.

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IT is well enough known (see, for instance, Borel, *Séries divergentes*, 1901, p. 164) how to deduce a Mittag-Leffler star expansion of a monogenic function from a given explicit expansion of $(1-z)^{-1}$ of the same character. Some readers may be interested in having such an expansion actually set forth; the method is, in part, merely a development of the original method of Runge (*Acta Mathematica*, VI, 1885, p. 237).

Given in the plane of the complex variable z any region of finite dimensions containing no point infinitely near to any point of the real axis from $z=1$ to $z=+\infty$, it will be shewn how to form a series of polynomials converging uniformly in this region and representing $(1-z)^{-1}$ therein.

Taking first a complex variable $\zeta, = \xi+i\eta$, enclose the points $\zeta=1$, $\zeta=1+c$, wherein c is an arbitrary real positive quantity, by a closed curve consisting of (i) the straight lines $\eta=\pm a$, from $\xi=1$ to $\xi=1+c$, the quantity a being real and positive and arbitrary (<1), (ii) a semicircle convex to the origin $\zeta=0$ satisfying the equation

$$(\xi-1)^2+\eta^2=a^2,$$

(iii) a semicircle concave to the origin, of equation $(\xi-1-c)^2+\eta^2=a^2$. Keeping c and a fixed for the present, take a positive integer r so that c/ra is less than unity, $=\sigma$ say; it is supposed that a is less than c, so that $r>1$; and take

$$c_0=1, \quad c_1=1+\frac{c}{r}, \quad c_2=1+\frac{2c}{r}, \quad \ldots, \quad c_r=1+c,$$

so that the segment from $\zeta=1$ to $\zeta=1+c$ is divided into r equal parts. If $n_1, n_2, \ldots, n_r$ be positive integers, the rational function

$$\frac{1}{1-\zeta}\left\{1-\left(\frac{c_1-1}{c_1-\zeta}\right)^{n_1}\right\}$$

is finite at $\xi = 1$, but has a pole of order n_1 at $\xi = c_1$; the rational function

$$\frac{1}{1-\xi}\left\{1-\left(\frac{c_1-1}{c_1-\xi}\right)^{n_1}\right\}\left\{1-\left(\frac{c_2-c_1}{c_2-\xi}\right)^{n_2}\right\}^{n_1}$$

is finite at $\xi = 1$, $\xi = c_1$, but has a pole of order $n_1 n_2$ at $\xi = c_2$; in general, writing

$$x_s = \left(\frac{c_s - c_{s-1}}{c_s - \xi}\right)^{n_s} \quad (s = 1, \ldots, r),$$

the rational function of ξ expressed by the product

$$U = \frac{1}{1-\xi}(1-x_1)(1-x_2)^{n_1}(1-x_3)^{n_1 n_2}\ldots(1-x_r)^{n_1 n_2 \ldots n_{r-1}},$$

has a pole only at $\xi = 1+c$, of order $n_1 n_2 \ldots n_r$.

The difference $$\frac{1}{1-\xi} - U$$

is of the form $(1-\xi)^{-1}P$, wherein, if $\rho_1, \rho_2, \ldots, \rho_k$ denote, in turn, x_1, x_2 repeated n_1 times,* x_3 repeated $n_1 n_2$ times, and so on, P is of the form

$$1-(1-\rho_1)(1-\rho_2)\ldots(1-\rho_k),$$

that is of the form $$\Sigma\rho_1 - \Sigma\rho_1\rho_2 + \Sigma\rho_1\rho_2\rho_3 - \ldots;$$

hence, if $r_i = |\rho_i|$, we have

$$\begin{aligned} |P| &\leqslant \Sigma r_1 + \Sigma r_1 r_2 + \Sigma r_1 r_2 r_3 + \ldots \\ &\leqslant (1+r_1)(1+r_2)\ldots(1+r_k) - 1. \end{aligned}$$

But, when ξ is without the closed curve above described round the segment from $\xi = 1$ to $\xi = 1+c$, we have

$$\left|\frac{1}{1-\xi}\right| < \frac{1}{a}, \qquad \left|\frac{c_s - c_{s-1}}{c_s - \xi}\right| < \frac{c/r}{a}, \qquad < \sigma.$$

Wherefore

$$\left|\frac{1}{1-\xi} - U\right| < \frac{1}{a}\left\{(1+\sigma^{n_1})(1+\sigma^{n_2})^{n_1}(1+\sigma^{n_3})^{n_1 n_2}\ldots(1+\sigma^{n_r})^{n_1 n_2 \ldots n_{r-1}} - 1\right\}.$$

Now, let ϵ be an arbitrary real positive quantity; take then μ a positive number, such that $e^{\mu} - 1 < \epsilon a$; suppose further that the positive integers $n_1, \ldots, n_r$ previously used are chosen, so that

$$\sigma^{n_1} < \frac{\mu}{1+\mu}, \quad \text{or} \quad \frac{\sigma^{n_1}}{1-\sigma^{n_1}} < \mu,$$

$$\sigma^{n_2} < \frac{1}{n_1}\sigma^{2n_1}, \quad \sigma^{n_3} < \frac{1}{n_1 n_2}\sigma^{3n_1}, \quad \ldots, \quad \sigma^{n_r} < \frac{1}{n_1 n_2 \ldots n_{r-1}}\sigma^{rn_1};$$

* So that $(1-x_2)^{n_1}$ is replaced by $(1-\rho_2)(1-\rho_3)\ldots(1-\rho_{n_1+1})$.

then, as $1+x < e^x$, the modulus of $(1-\xi)^{-1}-U$, is less than

$$\frac{1}{a}\{\exp(\sigma^{n_1}+n_1\sigma^{n_2}+n_1n_2\sigma^{n_3}+\ldots+n_1n_2\ldots n_{r-1}\sigma^{n_r})-1\},$$

and thus less than $\frac{1}{a}\{\exp(\sigma^{n_1}+\sigma^{2n_1}+\ldots+\sigma^{rn_1})-1\}$,

which is less than $\frac{1}{a}\left\{\exp\left(\frac{\sigma^{n_1}}{1-\sigma^{n_1}}\right)-1\right\}$;

and therefore less than the arbitrary quantity ϵ.

Now make a change of independent variable, putting

$$z = \frac{c\xi}{1+c-\xi},$$

equivalent with $$\xi = \frac{(1+c)z}{c+z};$$

thereby the points $\xi = 0$, $\xi = 1$, $\xi = 1+c$ become respectively the points $z = 0$, $z = 1$, $z = \infty$, z being real and positive so long as ξ is real and positive and $\xi < 1+c$; also

$$(1-z)^{-1} = \frac{c}{1+c}(1-\xi)^{-1}+\frac{1}{1+c};$$

the function U becomes a rational function of z with a pole only at $z = \infty$, that is, it is a polynomial in z; this we write in the form

$$U = \frac{1+c}{c}H-\frac{1}{c},$$

equivalent with $$H = \frac{c}{1+c}U+\frac{1}{1+c},$$

so that H is also a polynomial in z; thence

$$(1-z)^{-1}-H = \frac{c}{1+c}\{(1-\xi)^{-1}-U\}.$$

Further, the extreme points of the closed curve, namely,

$$\xi = 1-a,\ \eta = 0, \quad \text{and} \quad \xi = 1+c+a,\ \eta = 0,$$

become respectively, if $z = x+iy$, the points

$$x = c(1-a)/(c+a),\ y = 0, \quad \text{and} \quad x = -c(1+c+a)/a,\ y = 0;$$

as a approaches to zero the limiting positions of these are respectively $x = 1$, $y = 0$, and $x = -\infty$, $y = 0$. The lines $\eta = \pm a$ become por-

tions respectively of the two circles

$$(x+c)^2+y^2 = \pm \frac{c(c+1)}{a} y,$$

of which the former can be written

$$y = \frac{(x+c)^2}{2\mu} + \frac{(x+c)^4}{2\mu} \{\mu + [\mu^2-(x+c)^2]^{\frac{1}{2}}\}^{-2},$$

wherein $\mu = \frac{1}{2}c(c+1)/a$; thus, *for a given value of* x, by taking a sufficiently small, c being kept fixed, the ordinate y can be supposed arbitrarily small.

With these materials we can now recapitulate as follows:—Take in the plane of z any region of finite dimensions which does not include any point infinitely near to any point of the real axis from $z = 1$ to $z = +\infty$; take c an arbitrary real positive quantity (<1); take a, also real and positive, so small that the region enclosed as above by portions of the circles

$$(x+c)^2+y^2 = \pm c(c+1)\, y/a,$$

and curves passing through the points

$$z = c(1-a)/(c+a), \qquad z = -c(1+c+a)/a,$$

does not include any point of the originally given region; then take r so that $\sigma = c/ra$ is less than unity; and, taking an arbitrary real positive ϵ, take the positive integers $n_1, n_2, \ldots, n_r$ by the rules previously given. We can then form a polynomial in z, say H, of order $n_1 n_2 \ldots n_r$, such that throughout the originally given region

$$|\,(1-z)^{-1}-H\,| < \epsilon.$$

If, then, $\epsilon_1, \epsilon_2, \epsilon_3, \ldots$ be an aggregate of real positive numbers with zero as their limit, and the polynomial H corresponding to the case when ϵ is replaced by ϵ_m be denoted by H_m, the series of polynomials

$$H_1+(H_2-H_1)+(H_3-H_2)+\ldots,$$

whose sum to m terms is H_m, has a sum converging to $(1-z)^{-1}$ uniformly for the whole interior of the given region.

THEOREMS CONNECTED WITH MACLAURIN'S TEST FOR THE CONVERGENCE OF SERIES

By G. H. HARDY.

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1. In a paper recently published in these *Proceedings*,* Dr. Bromwich gave a very interesting extension of the theorem (usually but inaccurately ascribed to Cauchy) that, when $f(x)$ is a positive decreasing function of x, the series

(1) $$\sum^{\infty} f(n)$$

and the integral

(2) $$\int^{\infty} f(x)\,dx$$

converge or diverge together. Dr. Bromwich proved that, *if*

(i) $f(x)$ *is positive and tends steadily to zero;*

(ii) $\phi(x)$ *is positive and tends steadily to infinity,*

(iii) $\phi'(x)$ *tends steadily to zero,*

(iv) *the integral* $\int^{\infty} f(x)\,\phi'(x)\,dx$ *is convergent, then*

$$\int^{X} F(x)\,dx - \sum^{[X]} F(n),$$

where $F(x) = f(x)e^{i\phi(x)}$ *and* $[X]$ *denotes the integral part of* X, *tends to a finite limit as* $X \to \infty$.

Roughly, it may be said that what Dr. Bromwich proved amounted to this, that the relations between (1) and (2), as regards convergence or divergence, established by Maclaurin and Cauchy when $f(x)$ is positive and decreasing, still subsist when $f(x)$ is multiplied by an oscillatory factor of the type

$$\begin{matrix}\cos \\ \sin\end{matrix}\ \phi(x),$$

provided $\phi(x)$ *tends to infinity more slowly than* x.

* Vol. 6, pp. 327 *et seq.*

2. Dr. Bromwich's theorem may be generalised and its proof simplified as follows:—

THEOREM 1.—*If* (i) $F(x)$ *possesses a continuous derivative* $F'(x)$, (ii) $F(x) \to 0$ *as* $x \to \infty$, (iii) *the integral*

$$\int^{\infty} |F'(x)|\,dx$$

is convergent, then $$\int_0^X F(x)\,dx - \sum_1^{[X]} F(n)$$

tends to a finite limit, viz.,

$$J_1 = -\int_0^{\infty} (x-[x])\,F'(x)\,dx,$$

as $X \to \infty$.*

Let $$j_\nu = \int_{\nu-1}^{\nu} F(x)\,dx - F(\nu) = \int_{\nu-1}^{\nu} \{F(x)-F(\nu)\}\,\frac{d}{dx}\,(x-\nu+1)\,dx$$

$$= -\int_{\nu-1}^{\nu} (x-[x])\,F'(x)\,dx.$$

Then $$|j_\nu| \leqslant \int_{\nu-1}^{\nu} |F'(x)|\,dx.$$

Hence Σj_ν is convergent (absolutely) and the result follows immediately, since

$$\int_{[X]}^{X} F(x)\,dx \to 0.$$

3. In the case considered by Dr. Bromwich, we have

$$F'(x) = \{f'(x) + if(x)\,\phi'(x)\}\,e^{i\phi(x)},$$

and the convergence of $\int^{\infty} |F'(x)|\,dx$ follows immediately from that of

$$\int^{\infty} f'(x)\,dx, \qquad \int^{\infty} f(x)\,\phi'(x)\,dx.$$

Dr. Bromwich points out to me that the last inequality of § 2 may be established with equal simplicity by the method of argument adopted by himself (*loc. cit.*, p. 329). In fact

$$j_\nu = \int_{\nu-1}^{\nu} \{F(x)-F(\nu)\}\,dx = -\int_{\nu-1}^{\nu} dx \int_x^{\nu} F'(t)\,dt,$$

and so $$|j_\nu| \leqslant \int_{\nu-1}^{\nu} |F'(t)|\,dt.$$

So far as applications are concerned, there is practically nothing to choose

* It is not difficult to establish a similar theorem relating to double series and integrals.

between Dr. Bromwich's result and mine. For the sake of completeness I give an example in which my result applies and his does not. It is obtained by taking

$$F(x) = x^{-a} \exp\{ix^b \cos(\log x)\},$$

where $0 < b < a < 1$.

4. Either theorem enables us, as Dr. Bromwich shows, to determine very simply the behaviour, from the point of view of convergence, of many interesting series, such as $\Sigma(\log n)^p n^{-1-ai}$. But neither theorem is sufficiently general to deal with many other simple and interesting cases that naturally present themselves.

Consider, for example, the series

$$\Sigma \frac{e^{in^a}}{n^b},$$

where $0 < a < 1$, $b > 0$. The integral

$$\int^\infty \frac{e^{ix^a}}{x^b}\, dx$$

is convergent if $a+b > 1$, and it is natural to suppose that the same is the case with the series: and this is in fact true. But the conditions of Theorem 1 are satisfied only if $b > a$.

It is therefore desirable to investigate more general theorems.

5. It will easily be verified that, if we transform the expression j_ν of § 2 by a further integration by parts, we obtain *

$$\int_{\nu-1}^{\nu} \{f(x)+\tfrac{1}{2}f'(x)\}\, dx - f(\nu) = \tfrac{1}{2}\int_{\nu-1}^{\nu} (y^2-y) f''(x)\, dx,$$

or, say,

$$j_\nu^{(2)} = \tfrac{1}{2}\int_{\nu-1}^{\nu} (y^2-y) f''(x)\, dx,$$

where

$$y = x-[x].$$

And generally, if we denote by $\phi_k(x)$ the Bernoullian polynomial of degree k, so that †

$$\phi_1 = x, \quad \phi_2 = x(x-1), \quad \phi_3 = x(x-1)(x-\tfrac{1}{2}), \quad \phi_4 = x^2(x-1)^2, \quad \ldots,$$

* In what follows I revert to the notation $f(x)$, $f(n)$ instead of $F(x)$, $F(n)$.

† Bromwich, *Infinite Series*, p. 236.

we have

$$(3)\quad j_\nu^{(2k)} = \int_{\nu-1}^{\nu} \left\{ f(x) + \tfrac{1}{2} f'(x) + \frac{B_1}{2!} f''(x) - \frac{B_2}{4!} f^{(4)}(x) + \ldots + \frac{(-1)^{k-1} B_k}{2k!} f^{(2k)}(x) \right\} dx - f(\nu)$$

$$= -\frac{1}{(2k+1)!} \int_{\nu-1}^{\nu} \phi_{2k+1}(y) f^{(2k+1)}(x)\, dx$$

$$= \frac{1}{(2k+2)!} \int_{\nu-1}^{\nu} \phi_{2k+2}(y) f^{(2k+2)}(x)\, dx.$$

These formulæ are, in fact, merely a slight transformation of one of the standard forms of the "Euler-Maclaurin Sum Formula." *

From (3) we at once deduce

THEOREM 2.—*If either of the integrals*

$$\int^{\infty} |f^{(2k+1)}(x)|\, dx, \qquad \int^{\infty} |f^{(2k+2)}(x)|\, dx$$

is convergent, then

$$\int_0^{[X]} \left\{ f(x) + \tfrac{1}{2} f'(x) + \frac{B_1}{2!} f''(x) - \ldots \right\} dx - \sum_1^{[X]} f(\nu)$$

tends, as $X \to \infty$, *to a finite limit, viz., one of the integrals*

$$J_{2k+1} = -\frac{1}{(2k+1)!} \int_0^{\infty} \phi_{2k+1}(x - [x]) f^{(2k+1)}(x)\, dx,$$

$$J_{2k+2} = \frac{1}{(2k+2)!} \int_0^{\infty} \phi_{2k+2}(x - [x]) f^{(2k+2)}(x)\, dx.$$

If also $f(x)$ *and its first* $2k$ *derivatives tend to zero, as* $x \to \infty$, *then*

$$\int_0^X f(x)\, dx - \sum_1^{[X]} f(\nu)$$

tends, as $X \to \infty$, *to the limit*

$$\tfrac{1}{2} f(0) + \frac{B_1}{2!} f'(0) - \frac{B_2}{4!} f^{(3)}(0) + \ldots + \frac{(-1)^{k-1} B_k}{2k!} f^{(2k-1)}(0) + J_l,$$

where $l = 2k+1$ *or* $l = 2k+2$, *as the case may be.*

We have, in fact, merely to observe that the maximum of $|\phi_{2k+1}(y)|$ or $|\phi_{2k+2}(y)|$ depends on k only and not on $[X]$.

* Bromwich, *Infinite Series*, p. 239.

I have not thought it worth while to encumber this theorem (or the others which follow) with explicit statements as regards the continuity of such derivatives of $f(x)$ as figure in them. All such derivatives are, of course, supposed continuous: only conditions relating to their behaviour as $x \to \infty$ are in any way relevant to the problem. Sometimes, however, these conditions may be broken for $x = 0$ or some other value of x. Then the *formulæ* may need modification, but the application of the theorem to questions of convergence is not affected.

6. If $$f(x) = x^{-b} e^{ix^a},$$

where $0 < a < 1$, it is easy to see that

$$f^{(s)}(x) \sim (ia)^s x^{s(a-1)-b} e^{ix^a},$$

which yields an absolutely convergent integral if $s(a-1)-b < -1$ or

$$s > (1-b)/(1-a).$$

It is always possible to choose s so as to satisfy this condition: hence *the series and integral*

$$\int^{\infty} x^{-b} e^{ix^a} dx, \qquad \Sigma n^{-b} e^{in^a} \quad (0 < a < 1)$$

in all cases converge or oscillate together, i.e., converge if, and only if, $a+b > 1$.

If $b > a$ *is implied by* $a+b > 1$, *i.e.*, if $a \leqslant \frac{1}{2}$, but not if $a > \frac{1}{2}$, this result may be inferred from Theorem 1.

7. It is naturally suggested by the preceding results that it should be possible to prove theorems of a similar character, but relating to series and integrals that are only *summable* and not convergent. The integral

$$\int_0^{\infty} f(x)\,dx$$

is said to be summable $(C r)$, to sum S, if

$$\frac{r!}{x^r} \left(\int_0^x dt \right)^{r+1} f(t) \to S, \tag{4}$$

as $x \to \infty$.*

This definition is strictly analogous to Cesàro's definition of the sum

* Some properties of integrals which are summable $(C1)$ are proved by Hardy, *Quarterly Journal*, Vol. XXXV, p. 22; Moore, *Trans. Amer. Math. Soc.*, Vol. VIII, p. 299; Bromwich, *Math. Annalen*, Bd. LXV, S. 350. The above definition is an immediate generalisation of that adopted by these writers.

s of an oscillatory series Σa_ν, which may be put in the form

$$\frac{r!}{n^r}\left(\sum_{\nu=1}^{n}\right)^{r+1} a_\nu \to s. \tag{5}$$

Now, by a well known formula,*

$$\left(\int_0^x dt\right)^{r+1} f(t) = \frac{1}{r!}\int_0^x (x-t)^r f(t)\,dt.$$

Hence our definition is equivalent to

$$\int_0^x \left(1-\frac{t}{x}\right)^r f(t)\,dt \to S \tag{6}$$

or

$$\int_0^1 x f(xu)(1-u)^r\,du \to S. \tag{7}$$

And, when either of these forms is adopted, there is no longer any reason why r should be restricted to be an integer; we may suppose that r has any real value greater than -1.

Suppose, *e.g.*, that

$$f(x) = x^p e^{aix},$$

where a is real and not zero. Then the integral on the left-hand side of (7) is

$$x^{p+1}\int_0^1 u^p(1-u)^r e^{aixu}\,du.$$

Now it is known that, when $|x|$ is large,

$$\int_0^1 u^p(1-u)^r e^{aixu}\,du = \frac{\Gamma(p+1)}{(-aix)^{p+1}}(1+\epsilon)+\frac{\Gamma(r+1)}{(aix)^{r+1}}\,e^{aix}(1+\epsilon'),$$

where ϵ, ϵ' are small of the order $1/x$, and the many valued functions in the denominators have their values fixed by conventions explained precisely in the papers referred to in the foot-note below.†

It follows at once that

$$\int_0^\infty x^p e^{aix}\,dx$$

is summable (Cr), and has the sum

$$\Gamma(p+1)/(-ai)^{p+1}$$

if, and only if, $r > p$.

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* See, *e.g.*, Jordan, *Cours d'Analyse*, t. III, p. 59.

† Jacobsthal, *Math. Annalen*, Bd. LVI, S. 129; Barnes, *Trans. Camb. Phil. Soc.*, Vol. XX, p. 253; Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 3, p. 401.

8. Now M. Marcel Riesz has replaced* Cesàro's definition by

$$\sum_{\nu=1}^{n}\left(1-\frac{\nu}{n}\right)^{r} a_\nu \to s, \tag{8}$$

a definition precisely analogous in form to that adopted in equation (7) above for the value of the summable integral. This formal analogy renders it convenient to adopt this definition in the analysis which follows.

Before proceeding further, I may remark that I shall consider explicitly only the case in which r is a positive integer. This is the really interesting case. But the definition (8), like (7), applies equally well for non-integral values of r, and, in a good deal of the succeeding analysis, the restriction that r is integral is in no way necessary.

9. M. Riesz has indicated an important generalisation of the definition expressed by (8). Let $\lambda(x)$ be any function of x which tends steadily to infinity with x. Then we may define the sum s of the series Σa_n by the relation

$$\sum_{\nu=1}^{n}\left\{1-\frac{\lambda(\nu)}{\lambda(n)}\right\}^{r} a_\nu \to s, \tag{9}$$

which reduces to (8) if $\lambda(x)=x$. If this limit exists, we shall say that the series Σa_n is *summable (Rr) with sum-function* $\lambda(n)$. When $r=1$ this definition reduces to

$$s_n-\frac{1}{\lambda(n)}\sum_{\nu=1}^{n}\lambda(\nu)\,a_\nu \to s$$

or to

$$\frac{\mu_2 s_1+\mu_3 s_2+\ldots+\mu_n s_{n-1}}{\lambda(n)} \to s,$$

where $\mu_\nu=\lambda(\nu)-\lambda(\nu-1)$; which is the definition that I considered in a recent paper in these *Proceedings*.†

The corresponding definition for the integral is obviously

$$\int_0^x\left\{1-\frac{\lambda(t)}{\lambda(x)}\right\}^{r} f(t)\,dt \to S. \tag{10}$$

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* *Comptes Rendus*, July 5th, 1909. M. Riesz has established (as is easy enough when r is integral) the substantial equivalence of the two definitions.

† *Proc. London Math. Soc.*, Ser. 2, Vol. 8, p. 301.

If this limit exists, I shall say that the integral

$$\int_0^\infty f(x)\,dx$$

is *summable* (*Rr*), *with sum-function* $\lambda(x)$, *to sum* S.

If we put

$$\lambda(t) = T, \qquad \lambda(x) = X,$$

and denote by $\bar{\lambda}$ the function inverse to λ, we see that (10) is equivalent to

$$\int_0^X \left(1-\frac{T}{X}\right)^r f\{\bar{\lambda}(T)\}\,\bar{\lambda}'(T)\,dT.$$

That is to say,

$$\int^\infty f(x)\,dx$$

is summable (*Rr*), with sum-function $\lambda(x)$, if, and only if,

$$\int^\infty f\{\bar{\lambda}(x)\}\,\bar{\lambda}'(x)\,dx$$

is summable (*Rr*), with sum-function x. *The adoption of Riesz's general definition* instead of Cesàro's is, in the case of integrals, substantially equivalent to a change in the independent variable.*

Thus it is easy to prove that

$$\int^\infty (\log x)^p x^{-1+\alpha i}\,dx \tag{11}$$

is not summable (*Cr*) for any value of r. The substitution $x = e^y$, however, leads to the integral

$$\int^\infty y^p e^{\alpha i y}\,dy,$$

which is summable (*Cr*) for $r > p$. Hence (11) is summable (*Rr*), with sum-function $\log x$, for $r > p$. We shall see in a moment that the same is true of the series

$$\Sigma(\log n)^p n^{-1+\alpha i}.$$

10. We are now in a position to establish the following theorem :—

THEOREM 3.—*If* $f(x)$ *is subject to the conditions of Theorem* 1, *viz., that* $f(x) \to 0$ *and*

$$\int^\infty |f'(x)|\,dx$$

* I call the definition (10) Riesz's because he has doubtless used it as well as the definition (9). It would indeed be a plausible conjecture that M. Riesz was led to the definitions (8) and (9) through some consideration of integrals.

is convergent, and $\lambda(x)$ *is any function of* x *which has a continuous derivative and tends steadily to infinity with* x*, then*

$$\Sigma_n = \int_0^n \left\{1-\frac{\lambda(t)}{\lambda(n)}\right\}^r f(t)\,dt - \sum_1^n \left\{1-\frac{\lambda(\nu)}{\lambda(n)}\right\}^r f(\nu)$$

tends, as $n \to \infty$*, to the finite limit* J_1 *of Theorem* 1*, so that the summability of the series* $\Sigma f(\nu)$ *follows from that of the integral, the definitions of summability being Riesz's definitions of the* r*-th order, with sum-functions* $\lambda(x)$*,* $\lambda(n)$*.*

Let $$\psi(t) = \left\{1-\frac{\lambda(t)}{\lambda(n)}\right\}^r f(t).$$

Then $$\Sigma_n = \sum_1^n j_\nu,$$

where $$j_\nu = \int_{\nu-1}^{\nu} \psi(t)\,dt - \psi(\nu) = -\int_{\nu-1}^{\nu} (t-[t])\,\psi'(t)\,dt = u_\nu + v_\nu + w_\nu,$$

u_ν, v_ν, w_ν being defined by the equations

$$u_\nu = -\int_{\nu-1}^{\nu} (t-[t]) f'(t)\,dt, \tag{12}$$

$$v_\nu = \int_{\nu-1}^{\nu} (t-[t]) \left(1-\left\{1-\frac{\lambda(t)}{\lambda(n)}\right\}^r\right) f'(t)\,dt, \tag{13}$$

$$w_\nu = \frac{r}{\lambda(n)} \int_{\nu-1}^{\nu} \left\{1-\frac{\lambda(t)}{\lambda(n)}\right\}^{r-1} \lambda'(t) f(t)\,dt. \tag{14}$$

In the first place, $$|u_\nu| < \int_{\nu-1}^{\nu} |f'(t)|\,dt,$$

so that Σu_ν is convergent (absolutely). In the second place,

$$|w_\nu| < \frac{r}{\lambda(n)} \int_{\nu-1}^{\nu} \lambda'(t)\,|f(t)|\,dt.$$

But it is easy to prove that $\phi(t) \to 0$ involves

$$\frac{1}{\lambda(n)} \int_0^n \phi(t)\,\lambda'(t)\,dt \to 0 :*$$

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* This is an immediate generalisation from the well known case in which $\lambda(n) = n$. Choose N so that $|\phi| < \epsilon$ for $x \geqslant N$. Then

$$\frac{1}{\lambda(n)} \left|\int_0^n \phi\lambda'\,dt\right| < \frac{1}{\lambda(n)} \int_0^N |\phi\lambda'|\,dt + \frac{\epsilon}{\lambda(n)} \int_N^n \lambda'(t)\,dt < \frac{1}{\lambda(n)} \int_0^N |\phi\lambda'|\,dt + \epsilon < 2\epsilon,$$

if n is large enough.

and so $$\sum_1^n w_\nu \to 0.$$

Finally, $$\left|\sum_1^n v_\nu\right| < \int_0^n \left(1-\left\{1-\frac{\lambda(t)}{\lambda(n)}\right\}^r\right)|f'(t)|\,dt.$$

Let $$\int_t^\infty |f'(u)|\,du = \Phi(t).$$

Then, integrating by parts,

$$\left|\sum_1^n v_\nu\right| < -\Phi(n)+\left(1-\left\{1-\frac{\lambda(0)}{\lambda(n)}\right\}^r\right)\Phi(0)$$

$$-\frac{r}{\lambda(n)}\int_0^n \left\{1-\frac{\lambda(t)}{\lambda(n)}\right\}^{r-1}\lambda'(t)\,\Phi(t)\,dt.$$

The first two terms evidently tend to zero as $n \to \infty$. The last is numerically less than

$$\frac{r}{\lambda(n)}\int_0^n \lambda'(t)\,|\Phi(t)|\,dt,$$

and so also tends to zero.

It follows that $$\Sigma_n \to \sum_1^\infty u_\nu = -\int_0^\infty (t-[t])\,f'(t)\,dt,$$

and the proof of Theorem 3 is accordingly completed.

11. A simple example of the theorem just proved is afforded by the case in which

$$f(x) = (\log x)^p\, x^{-1+ai}.$$

Here $$f'(x) \sim (ai-1)(\log x)^p\, x^{-2+ai},$$

and the conditions of the theorem are certainly satisfied. It follows from § 9 that the series

$$\Sigma(\log n)^p\, n^{-1+ai}$$

is summable (*Rr*), with sum function $\log n$, if, and only if, $r > p$.

12. The question now arises as to whether we can prove a still more general theorem related to Theorem 3 as Theorem 2 is related to Theorem 1.

I shall content myself with proving this in two special cases, viz., (1) when

$$\lambda(x) = x$$

(so that we are using definitions substantially equivalent to Cesàro's), and (2) when $\lambda(x)$ is arbitrary and $k = 0$. The absolutely general case appears to require rather elaborate preliminaries.

13. The formula (3) of § 5, applied to the function

$$\psi(t) = \left(1-\frac{t}{n}\right)^r f(t),$$

gives

$$(15)\quad \int_{\nu-1}^{\nu} \left\{ \psi(t)+\tfrac{1}{2}\psi'(t)+\frac{B_1}{2!}\psi''(t)-\ldots+\frac{(-1)^{k-1}B_k}{2k!}\psi^{(2k)}(t)\right\} dt-\psi(\nu)$$

$$= \frac{1}{(2k+2)!}\int_{\nu-1}^{\nu} \phi_{2k+2}(t-[t])\,\psi^{(2k+2)}(t)\,dt.$$

Suppose now that, as in Theorem 2,

$$\int^{\infty} |f^{(2k+2)}(t)|\,dt,$$

and is convergent, and also that $f^{(2k+1)}(t) \to 0$. Then we can prove that

$$\sum_1^n \int_{\nu-1}^{\nu} \phi_{2k+2}(t-[t])\,\psi^{(2k+2)}(t)\,dt \to \int_0^{\infty} \phi_{2k+2}(t-[t])\,f^{(2k+2)}(t)\,dt.$$

We have, in fact,

$$\psi^{(2k+2)}(t) = f^{(2k+2)}(t)-\left\{1-\left(1-\frac{t}{n}\right)^r\right\} f^{(2k+2)}(t)$$

$$+\sum_{s=1}^{\sigma}(-1)^s\,s!\binom{2k+2}{s}\binom{r}{s}n^{-s}\left(1-\frac{t}{n}\right)^{r-s} f^{(2k+2-s)}(t),$$

where σ is the lesser of the numbers $2k+2$ and r.

We may therefore write

$$\int_{\nu-1}^{\nu} \phi_{2k+2}(t-[t])\,\psi^{(2k+2)}(t)\,dt = u_\nu+v_\nu+\sum_{s=1}^{\sigma} w_\nu^{(s)},$$

where $$u_\nu = \int_{\nu-1}^{\nu} \phi_{2k+2}(t-[t])\, f^{(2k+2)}(t)\, dt,$$

$$v_\nu = -\int_{\nu-1}^{\nu} \phi_{2k+2}(t-[t]) \left\{1-\left(1-\frac{t}{n}\right)^r\right\} f^{(2k+2)}(t)\, dt,$$

$$w_\nu^{(s)} = C_s n^{-s} \int_{\nu-1}^{\nu} \phi_{2k+2}(t-[t]) \left(1-\frac{t}{n}\right)^{r-s} f^{(2k+2-s)}(t)\, dt,$$

C_s being a number dependent only on k, r, s, and not upon ν or n.

Since $$|u_\nu| < M \int_{\nu-1}^{\nu} |f^{(2k+2)}(t)|\, dt,$$

where M is the maximum of $\phi_{2k+2}(y)$ for $0 < y < 1$, it follows that Σu_ν is convergent (absolutely).

Again, $$\left|\sum_1^n v_\nu\right| < M \int_0^n \left\{1-\left(1-\frac{t}{n}\right)^r\right\} |f^{(2k+2)}(t)|\, dt.$$

Let $$\Phi(t) = \int_t^\infty |f^{(2k+2)}(u)|\, du.$$

Then $$\int_0^n \left\{1-\left(1-\frac{t}{n}\right)^r\right\} |f^{(2k+2)}(t)|\, dt = -\Phi(n) + \frac{r}{n}\int_0^n \left(1-\frac{t}{n}\right)^{r-1} \Phi(t)\, dt.$$

The first term on the right-hand side has the limit zero. The second is less than

$$\frac{r}{n}\int_0^n \Phi(t)\, dt,$$

and so also tends to zero. It follows that

(16) $$\sum_1^n v_\nu \to 0.$$

Finally, $$\left|\sum_1^n w_\nu^{(s)}\right| < K n^{-s} \int_0^n |f^{(2k+2-s)}(t)|\, dt,$$

where K is independent of n. I shall prove in the next section that the expression on the right-hand side tends to zero as $n \to \infty$, for $s = 1, 2, \ldots, \sigma$. It will then follow that

(17) $$\sum_1^n w_\nu^{(s)} \to 0 \quad (s = 1, 2, \ldots, \sigma),$$

and from (16) and (17) will follow the truth of the assertion made at the beginning of this section.

14. Since $f^{(2k+1)}(t) \to 0$, it follows at once that

$$\frac{1}{t}\int_0^t |f^{(2k+1)}(u)|\,du \to 0,$$

as $t \to \infty$. Hence

$$|f^{(2k)}(t)| \leqslant |f^{(2k)}(0)| + \int_0^t |f^{(2k+1)}(u)|\,du = t\epsilon_t,$$

where $\epsilon_t \to 0$. Hence

$$\frac{1}{t^2}\int_0^t |f^{(2k)}(u)|\,du \to 0,$$

and from this we deduce as above that

$$|f^{(2k-1)}(t)| = t^2\epsilon_t,$$

where $\epsilon_t \to 0$. It is evident that a mere repetition of this argument is sufficient to prove what we require.

It is therefore proved that the sum from $\nu = 1$ to $\nu = n$ of the right-hand side of the equation (15) tends, as $n \to \infty$, to the limit

$$J_{2k+2} = \frac{1}{(2k+2)!}\int_0^\infty \phi_{2k+2}(t-[t])\,f^{(2k+2)}(t)\,dt.$$

15. We have now to consider the left-hand side of (15), which is

$$\Sigma_n + \tfrac{1}{2}\{\psi(n)-\psi(0)\} + \frac{B_1}{2!}\{\psi'(n)-\psi'(0)\} - \frac{B_2}{4!}\{\psi^{(3)}(n)-\psi^{(3)}(0)\} + \ldots$$

$$+ \frac{(-1)^{k-1}B_k}{2k!}\{\psi^{(2k-1)}(n)-\psi^{(2k-1)}(0)\},$$

where $$\Sigma_n = \int_0^n \left(1-\frac{t}{n}\right)^r f(t)\,dt - \sum_1^n \left(1-\frac{\nu}{n}\right)^r f(\nu).$$

Now $$\psi^{(s)}(t) = \sum_{\kappa=0}^{\lambda} (-1)^\kappa \kappa! \binom{s}{\kappa}\binom{r}{\kappa} n^{-\kappa}\left(1-\frac{t}{n}\right)^{r-\kappa} f^{(s-\kappa)}(t),$$

where λ is the lesser of the numbers r, s. From this it follows at once that

(18) $$\psi^{(s)}(0) = f^{(s)}(0) + \epsilon_n,$$

where $\epsilon_n \to 0$ as $n \to \infty$. Also

$$\psi^{(s)}(n) = 0 \quad (r > s), \tag{19}$$

since, if $r > s$, every term in $\psi^{(s)}(n)$ contains a factor which vanishes for $t = n$. On the other hand, if $r \leqslant s$, there is one term in $\psi^{(s)}(n)$ which does not vanish for $t = n$, viz., that for which $\kappa = r$, and

$$\psi^{(s)}(n) = (-1)^r r! \binom{s}{r} n^{-r} f^{(s-r)}(n) \quad (r \leqslant s). \tag{20}$$

From (18), (19), and (20) it follows that the left-hand side of (15) may be expressed in the form

$$\Sigma_n + \epsilon_n - \tfrac{1}{2} f(0) - \frac{B_1}{2!} f'(0) + \frac{B_2}{4!} f^{(3)}(0) - \ldots - \frac{(-1)^{k-1} B_k}{2k!} f^{(2k-1)}(0) + \chi(n),$$

where $$\chi(n) = (-1)^r r!\, n^{-r} \Sigma \frac{(-1)^{p-1} B_p}{2p!} \binom{2p-1}{r} f^{(2p-1-r)}(n),$$

the summation extending to all the values of p (if any) which satisfy the inequalities

$$r \leqslant 2p-1 \leqslant 2k-1.$$

We therefore assume, in addition to the conditions already imposed upon f, that

$$x^{-r} f^{(2p-1-r)}(x) \to 0,$$

for $r \leqslant 2p-1 \leqslant 2k-1$. Thus, if $r = 1$, we must have

$$f(x)/x \to 0, \quad f''(x)/x \to 0, \quad \ldots, \quad f^{(2k-2)}(x)/x \to 0;$$

if $r = 2k-1$, we must have

$$f(x)/x^{2k-1} \to 0;$$

while, if $r \geqslant 2k$, no conditions of this type are needed. If, *e.g.*, $k = 1$, the only case in which one of these conditions is required is when $r = 1$, the condition then being

$$f(x)/x \to 0.$$

16. We have thus proved—

THEOREM 4.—*If*

(i) $\int^{\infty} |f^{(2k+2)}(x)|\, dx$ *is convergent,*

(ii) $f^{2k+1}(x) \to 0$,

(iii) $f^{(2p-1-r)}(x)/x^r \to 0$

for such values of p, if any, as satisfy the inequalities

$$r \leqslant 2p-1 \leqslant 2k-1,$$

then

$$\int_0^n \left(1-\frac{t}{n}\right)^r f(t)\,dt - \sum_1^n \left(1-\frac{\nu}{n}\right)^r f(\nu)$$

tends, as $n \to \infty$, to the limit

$$\tfrac{1}{2}f(0) + \frac{B_1}{2!}f'(0) - \frac{B_2}{4!}f^{(3)}(0) + \ldots$$

$$+ \frac{(-1)^{k-1}B_k}{2k!}f^{(2k-1)}(0) + \frac{1}{(2k+2)!}\int_0^\infty \phi_{2k+2}(t-[t])\,f^{(2k+2)}(t)\,dt,$$

so that the summability (Rr), with sum-function n, of the series $\Sigma f(n)$ follows from that of the integral $\int f(x)\,dx$.

In the simplest case, that of $k = 0$, the theorem takes the form

THEOREM 5.—*If* (i) $\int^\infty |f''(x)|\,dx$ *is convergent,* (ii) $f'(x) \to 0$, *then*

$$\int_0^n \left(1-\frac{t}{n}\right)^r f(t)\,dt - \sum_1^n \left(1-\frac{\nu}{n}\right)^r f(\nu) \to \tfrac{1}{2}f(0) + \frac{1}{2!}\int_0^\infty \phi_2(t-[t])\,f''(t)\,dt,$$

if $r > 0$; and if $r = 0$, the result is still true, provided only

$$f(x) \to 0.$$

17. Let us consider, for example, the case in which

$$f(x) = x^{-b}e^{ix^a} \quad (0 < a < 1).$$

Then

$$f^{(s)}(x) \sim (ia)^s\, x^{s(a-1)-b}\, e^{ix^a},$$

which tends to zero, if $s(a-1)-b < 0$, and possesses an absolutely convergent integral, if $s(a-1)-b < -1$. This last condition will be satisfied for $s = 2k+2$, if

$$2k+2 > (1-b)/(1-a).$$

This is certainly so for sufficiently large values of k.

The condition that $f^{(2k+1)}(x) \to 0$ requires

$$2k+1 > -b/(1-a);$$

and, as $1/(1-a) > 1$, this is a consequence of the former condition.

The condition $$f^{(2p-1-r)}(x)/x^r \to 0,$$

reduces to $$(2p-1-r)(a-1)-b-r < 0.$$

This is *least* likely to be satisfied when p has its smallest possible value, viz., that given by $r = 2p-1$. It then reduces to

$$b+r > 0.$$

But the condition $b+r > 0$ is certainly a *necessary* condition for the summability of

$$\Sigma n^{-b} e^{in^a},$$

by Cesàro's r-th mean.* Hence there is no loss of generality involved in this set of conditions.

We can now apply Theorem 4, and we see that the problem of the summability of the series is reduced to the corresponding problem for the integral

$$\int x^{-b} e^{ix^a} dx.$$

We have therefore only to determine whether

$$\int_0^n \left(1-\frac{t}{n}\right)^r \frac{e^{it^a}}{t^b} dt$$

tends to a limit as $n \to \infty$. Putting

$$t = nu^{1/a} = (Nu)^{1/a},$$

we obtain $$\frac{1}{a} N^{(1-b)/a} \int_0^1 (1-u^{1/a})^r u^{[(1-b)/a]-1} e^{iNu} du.$$

Now it may easily be proved, by a slight modification of the argument used in establishing the asymptotic formula quoted in § 7, that

$$\int_0^1 (1-u^{1/a})^r u^{[(1-b)/a]-1} e^{iNu} du = \frac{\Gamma\{(1-b)/a\}}{(-iN)^{(1-b)/a}} (1+\epsilon) + \frac{\Gamma(r+1)}{a^r (iN)^{r+1}} e^{iN} (1+\epsilon'),$$

* Bromwich, *Infinite Series*, p. 318.

where ϵ and ϵ' are of order $1/N$ when N is large. It follows that the integral

$$\int_0^\infty \frac{e^{it^a}}{t^b}\,dt$$

is summable (Cr), and has the sum

$$\frac{1}{a}\Gamma\left(\frac{1-b}{a}\right)\Big/(-i)^{(1-b)/a},$$

if, and only if, $r+1>(1-b)/a$; *i.e.*, if

$$b+(r+1)\,a>1.*$$

And so the same is true of the series: that is to say, *the series and integral*

$$\sum^\infty n^{-b}e^{in^a}, \qquad \int^\infty x^{-b}e^{ix^a}\,dx \quad (0<a<1)$$

are both convergent if $a+b>1$, *both summable* $(R1)$ *if* $2a+b>1$, *both summable* $(R2)$ *if* $3a+b>1$, *and so on, with sum function n.*

18. The last question which arises is as to whether there is an analogue of Theorem 4, when Riesz's more general definition is adopted. The answer is, of course, in the affirmative; but, for the reason stated in § 12, I content myself with considering the simplest case, the analogue of Theorem 5.

THEOREM 6.—*If* (i) $\int^\infty |f''(x)|\,dx$ *is convergent,* (ii) $f'(x)\to 0$, (iii) $\lambda(x)$ *is a steadily increasing function of* x, *tending to infinity with* x, *and possessing a continuous derivative,* (iv) $\{\lambda'(x)/\lambda(x)\}f(x)\to 0$, *then*

$$\int_0^n \left\{1-\frac{\lambda(t)}{\lambda(x)}\right\}^r f(t)\,dt-\sum_1^n \left\{1-\frac{\lambda(\nu)}{\lambda(n)}\right\}^r f(\nu)$$

tends, as $n\to\infty$, *to the limit*

$$\tfrac{1}{2}f(0)+\tfrac{1}{2}\int_0^\infty \phi_2(t-[t])\,f''(t)\,dt,$$

if $r>0$; *and the conclusion remains valid for* $r=0$, *provided* $f(x)\to 0$.

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* Of course we suppose $b<1$ in order that the integral should converge at the lower limit: this is quite irrelevant to the main issue. If $b>1$, the integral is absolutely convergent as regards the upper limit.

We have, in fact, if Σ_n denotes the expression whose limit we are considering,

$$\Sigma_n+\tfrac{1}{2}\{\psi(n)-\psi(0)\} = \frac{1}{2!}\int_0^n \phi_2(t-[t])\psi''(t)\,dt,$$

where
$$\psi(t) = \left\{1-\frac{\lambda(t)}{\lambda(n)}\right\}^r f(t).$$

Clearly $\psi(n) = 0$, unless $r = 0$, in which case we impose the additional condition that

$$\psi(x) = f(x) \to 0.$$

Again,

$$\psi''(t) = \left\{1-\frac{\lambda(t)}{\lambda(n)}\right\}^r f''(t) - \frac{2r\lambda'(t)}{\lambda(n)}\left\{1-\frac{\lambda(t)}{\lambda(n)}\right\}^{r-1} f'(t)$$

$$+\frac{r(r-1)\{\lambda'(t)\}^2}{\{\lambda(n)\}^2}\left\{1-\frac{\lambda(t)}{\lambda(n)}\right\}^{r-2} f(t)$$

$$-\frac{r\lambda''(t)}{\lambda(n)}\left\{1-\frac{\lambda(t)}{\lambda(n)}\right\}^{r-1} f(t).$$

The reader who has followed the arguments of the preceding paragraphs with care (especially those of §§ 10, 13) will easily convince himself that everything reduces to proving that $f'(x) \to 0$ involves

$$(a)\quad \frac{1}{\lambda(x)}\int_0^x \lambda'(t)f'(t)\,dt \to 0,$$

$$(b)\quad \frac{1}{\{\lambda(x)\}^2}\int_0^x \{\lambda'(t)\}^2 f(t)\,dt \to 0,$$

$$(c)\quad \frac{1}{\lambda(x)}\int_0^x \lambda''(t)f(t)\,dt \to 0.$$

Of these relations the first has already been proved. Again,

$$\frac{1}{\lambda(x)}\int_0^x \lambda''(t)f(t)\,dt = \frac{\lambda'(x)}{\lambda(x)}f(x) - \frac{\lambda'(0)}{\lambda(x)}f(0) - \frac{1}{\lambda(x)}\int_0^x \lambda'(t)f'(t)\,dt,$$

and so also tends to zero if the condition

$$\frac{\lambda'(x)}{\lambda(x)}f(x) \to 0$$

is satisfied, as we have supposed. Also, if this condition is satisfied, we can find X, so that

$$|f(x)| < \epsilon\lambda(x)/\lambda'(x) \quad (x \geqslant X).$$

Then $$\frac{1}{\{\lambda(x)\}^2}\left|\int_0^x \{\lambda'(t)\}^2 f(t)\,dt\right| < \frac{1}{\{\lambda(x)\}^2}\int_0^X \{\lambda'(t)\}^2 |f(t)|\,dt$$

$$+\frac{\epsilon}{\{\lambda(x)\}^2}\int_X^x \lambda(t)\lambda'(t)\,dt.$$

The second term is less than $\frac{1}{2}\epsilon$; and by choosing x large enough, when X is fixed, we can ensure that the first term is also less than $\frac{1}{2}\epsilon$. The relations (a), (b), (c) are accordingly established and Theorem 6 is proved.

NOTES ON THE THEORY OF THE CUBIC SURFACE

By H. F. Baker.

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The present paper consists of two parts; to the second part (p. 177) which I have named Appendix, largely written this year, I have prefixed some introductory remarks explaining my indebtedness to others; for the first part, §§ 1–13, a similar apology seems necessary. It was put together mostly in the summer vacations of 1908, 1909, partly for lectures, and then, as I found the material to be so scattered, for my own use; in giving the lectures it was necessary to start from the stage reached by Salmon in the *Solid Geometry* and the *Higher Plane Curves*, and it seems desirable in the present abbreviated account to take the same starting point. Some of the less known results were brought before the Society in January, 1910. The method of presentment is primarily analytical, which experience suggests may be of advantage to readers not already familiar with the matter; for a much more attractive presentment of some of the leading ideas the reader is referred to a paper of Miss Long.* The following details may enable readers to judge how far the present account contains material not already known to them.

§ 1 briefly reproduces the well known method of regarding a plane quartic as an envelope of conics. § 2 discusses the Steiner systems of bitangents of a plane quartic, giving a direct proof of the theorem that the six centres lie on a conic; this theorem is curiously not mentioned by Salmon, though it is clear enough from the theory of Kummer's quartic surface. The section also gives an obvious generalisation of the theorem. § 3 gives Geiser's proof of the same result. § 4 considers the Steiner systems as consisting of tangents of a curve of class three; the latter part of the section, which was written in April of this year, shews that in the elliptic representation of this curve of class three the twelve bitangents of a Steiner system may be represented by six arguments of zero sum and six others which are corresponding tangents to these, all of the same system, and are therefore represented by six elliptic arguments differing from the former by the same half period. This fact was discovered in

* Printed below in these *Proceedings*.

generalisation of a theorem mentioned to me (see § 11, last example). § 5 briefly gives Geiser's method of obtaining the plane quartic curve from a cubic surface; it was this deduction, with a view to simplifying the theory of the bitangents, which was at first the object of the paper. § 6 deals with the systematic determination of the lines of a cubic surface, and the nomenclature for these lines by means of the binary combinations of eight symbols; though the matter in general is well enough known, I am not able to give a reference for the rule of thumb here found for the intersection of two lines of the surface. § 7 gives the geometrical interpretation of the seven primary lines from which the notation of § 6 is derived, shewing that the other twenty lines are rationally determinable from these. It is of interest also as leading (in § 11) to an intuitive proof of Aronhold's determination of a plane quartic from seven given bitangents. It was one of the points explained to the Society in January of this year. § 8 shews how Riemann's equations for the bitangents of a plane quartic may be deduced at this stage. § 9 discusses in detail the correspondence between the lines of a cubic surface and the bitangents of a plane quartic, and gives a proof, which is new, so far as I know, that there are in effect only two ways in which a Steiner system of bitangents arises from lines of the cubic surface. The theorem is referred to by Geiser (*Crelle*, Bd. LXXIII, p. 377) with indication of a different mode of proof which I have not followed out in detail. As Geiser's paper contains, in its final paragraph, a similar mention of another theorem, which is not true, it seemed worth while to give a proof here. A particular consequence (given in § 11, third paragraph) was set as an example in Part II of the Mathematical Tripos, 1909. § 10 carries on the reasoning of § 9, introducing the double sixes of the cubic surface. § 11 gives a few examples of the consequences of the point of view, and in particular the theorem that the six transversals from any point of the cubic surface, one to each of the opposite line pairs of a double six, lie on a quadric cone. This theorem, of which I have not seen mention, arises here as an obvious corollary from the fact that the six centres of a Steiner system of bitangents lie upon a conic; but it appears an interesting theorem, and there is good evidence that it is not well known. It suggests the problem of finding the locus of a point from which the six transversals to six pairs of non-intersecting lines may be on a quadric cone. This locus, of the 24th degree in general, when the six lines are a double six of a cubic surface, apparently includes not only the cubic surface, but also its reciprocal, of the 12th degree. § 12 gives a proof of an interesting theorem incidentally given by Reye, after the method of Beltrami. This method of Beltrami, applied to prove the canonical form for the cubic surface due to Sylvester, I first heard of

in conversation with Mr. Richmond, in February, 1909; a similar argument is here applied to the rational curve of order n in space of n dimensions, but I do not know that the theorem obtained is new. § 13 gives a geometrical interpretation of the problem of finding the lines of a cubic surface of which the Sylvester equation is known, which is certainly curious, but may not be very useful.

1. *The Quartic Curve as an Envelope of Conics.*

If $y=0$, $z=0$ be any two bitangents of a plane quartic curve, the equation of the curve necessarily has a form

$$\phi^2-yz\psi=0,$$

where ϕ, ψ are homogeneous of the second degree in y, z, and a third coordinate x. This is identically the same as

$$(\phi+\sigma yz)^2-yz(\sigma^2yz+2\sigma\phi+\psi)=0,$$

or as $$[\sigma_1\sigma_2yz+(\sigma_1+\sigma_2)\phi+\psi]^2-[\sigma_1^2yz+2\sigma_1\phi+\psi][\sigma_2^2yz+2\sigma_2\phi+\psi]=0,$$

wherein σ, σ_1, σ_2 are arbitrary. The general conic through the four points of contact of the curve with $y=0$, $z=0$ is $\phi+\sigma yz=0$; this meets the curve again in four points; these four points are points of contact of the curve with the conic $\sigma^2yz+2\sigma\phi+\psi=0$; the eight points of contact of any two conics of this form, say for $\sigma=\sigma_1$, $\sigma=\sigma_2$, lie upon another conic, namely,

$$\sigma_1\sigma_2yz+(\sigma_1+\sigma_2)\phi+\psi=0.$$

If $\chi=0$ be any conic other than $\phi=0$, $\psi=0$, since the substitution

$$\rho=(a+b\sigma)/(1+\sigma)$$

reduces $$\rho^2\chi+2\rho\phi+\psi=0$$

to the form

$$\sigma^2(a^2\chi+2a\phi+\psi)+2\sigma[ab\chi+(a+b)\phi+\psi]+b^2\chi+2b\phi+\psi=0,$$

we see that no generality is lost or gained by regarding the quartic as the envelope of conics of the form

$$\rho^2\chi+2\rho\phi+\psi=0,$$

instead of $$\sigma^2yz+2\sigma\phi+\psi=0,$$

and so having the form $$\phi^2=\psi\chi,$$

or conversely, by choosing a, b, so that

$$a^2\chi+2a\phi+\psi=yz, \qquad b^2\chi+2b\phi+\psi=xP,$$

as having the form $\Phi^2 = xyzP,$

where $P = 0$ is a straight line and Φ denotes a conic $ab\chi+(a+b)\phi+\psi$.

If c be a third constant such that

$$c^2\chi+2c\phi+\psi = \eta\xi,$$

the irrational equations of the quartic

$$ab\chi+(a+b)\phi+\psi = \sqrt{yz}\sqrt{xP}, \quad ac\chi+(a+c)\phi+\psi = \sqrt{yz}\sqrt{\eta\xi},$$

coupled with $a^2\chi+2a\phi+\psi = yz,$

give, as a form for the quartic,

$$(b-c)\sqrt{yz}+(c-a)\sqrt{xP}+(a-b)\sqrt{\eta\xi} = 0.$$

2. *The Steiner Systems; Direct Proof that their Centres lie on a Conic.*

There are, in fact, six values of ρ for which the conic

$$\rho^2\chi+2\rho\phi+\psi = 0$$

breaks up into a pair of straight lines; the twelve pairs of bitangents of the quartic curve $\phi^2 = \psi\chi$ so obtained will be said to form a *Steiner system.* Such a system has various properties; we have just seen that there is on the quartic a linear relation connecting the square roots of the products of any three of these line pairs; we prove now that the six points of intersection of the various pairs lie on a conic. The fact is not referred to by Salmon (*Higher Plane Curves*, 3rd edition, 1879), but is mentioned and proved as below (in § 3) by Geiser, *Crelle*, Bd. LXXIII. A direct proof which is quite simple is as follows.

If $$(a, b, c, f, g, h\,\text{⨞}\,x, y, z)^2 = 0$$

represent a pair of straight lines intersecting in (x, y, z), the three equations such as $ax+hy+gz = 0$ are satisfied, and therefore also the equations

$$x^2/A = y^2/B = z^2/C = yz/F = zx/G = xy/H,$$

where $$A = bc-f^2, \quad F = gh-af, \quad \ldots;$$

if, then, $(a, b, c, f, g, h\,\text{⨞}\,x, y, z)^2$

$$= \rho^2(a_1, b_1, c_1, f_1, g_1, h_1\,\text{⨞}\,x, y, z)^2+\rho(a_2, b_2, c_2, f_2, g_2, h_2\,\text{⨞}\,x, y, z)^2 + (a_3, b_3, c_3, f_3, g_3, h_3\,\text{⨞}\,x, y, z)^2,$$

and A_i denote what A, a polynomial of degree 4 in ρ, becomes when for ρ is substituted one of the six values for which this conic is two straight

lines, the condition that the six points of intersection of the pairs of a Steiner system should lie on a conic is the vanishing of a determinant of six rows and columns of which one row is

$$A_i, B_i, C_i, F_i, G_i, H_i.$$

This determinant, however, regarded as a polynomial of the fourth degree in a variable parameter ρ_1 manifestly vanishes for the five values of ρ_1 expressed by $\rho_1 = \rho_2$, $\rho_1 = \rho_3$, ... It is thus identically zero in regard to ρ_1. Thus the determinantal condition is satisfied and the proposition follows.

It may be remarked that this proof is capable of immediate generalisation to $2r$ dimensions; if $\phi_1, \phi_2, \ldots, \phi_{r+2}$ be quadric polynomials homogeneous in $2r+1$ variables, there are $(2r+1)(r+1)$ values of λ for which the $(2r-1)$-fold

$$\phi_1\lambda^{r+1}+\phi_2\lambda^r+\ldots+\phi_{r+2}=0$$

has a node, namely, a point such that the line (one-fold) joining it to any other point of the quadric lies entirely upon the quadric. The theorem is that the quadric $(2r-1)$-fold drawn through all but one of these nodes contains the remaining node.

3. *Geometrical Proof of the same Result.*

The same result may be obtained otherwise thus, as indicated by Geiser. It is well known that any three conics χ, 2ϕ, ψ may be regarded as first polars of a definite cubic curve; a proof of this, with reference to original authorities is given, for instance, in the *Camb. Phil. Trans.*, Vol. xv, p. 92; the conics $\rho^2\chi+2\rho\phi+\psi=0$ are then first polars in regard to this cubic f of points $x'=\rho^2$, $y'=\rho$, $z'=1$ lying upon the conic $x'z'-y'^2=0$; these conics $\rho^2\chi+2\rho\phi+\psi=0$ are straight line-pairs for the six points (x', y', z'), where the conic $x'z'-y'^2=0$ cuts the Hessian H, of f, the intersection (x_0, y_0, z_0) of such a line-pair being a corresponding point of (x', y', z') on H; so that these six points (x_0, y_0, z_0) also lie upon a conic.

4. *The Steiner System as Tangents of a Curve of Class Three.*

Salmon proves (*loc. cit.*, pp. 150, 226) that the twelve bitangents of a Steiner system all touch a curve of the third class, the Cayleyan Γ, of the Hessian H, and that, if two of the bitangents forming a pair be taken, and their points of contact with the quartic curve be joïned both directly and transversely, the four joining lines also touch the Cayleyan Γ, and the two intersections of each pair of joining lines lie on the Hessian H.

To this we can add a further remark of interest. If four points upon a cubic curve, say P, Q, P', Q', have the property that the other point pair of the quadrangle of which opposite corners are the point pairs PP' and QQ', namely, the points of intersection $(PQ, P'Q')$, $(PQ', P'Q)$, also lie upon the cubic curve, then P', Q' are corresponding points to P, Q respectively, of the same kind. This is obvious analytically: representing P, Q, P', Q' by the elliptic arguments u, v, $u+a$, $v+\beta$, the data are that

$$u+v \equiv u+v+a+\beta, \qquad u+v+\beta = u+v+a\,;$$

these give $a \equiv \beta \equiv$ half-period. Reciprocally, if four tangents l, m, l', m', of a curve of class three be such that the other line-pair through their four intersections are also tangents of the class cubic, then l' corresponds to l and m' to m in the same kind. Returning then to the figure of § 3, if

$$x_1'\frac{\partial f}{\partial x}+y_1'\frac{\partial f}{\partial y}+z_1'\frac{\partial f}{\partial z}=0, \qquad x_2'\frac{\partial f}{\partial x}+y_2'\frac{\partial f}{\partial y}+z_2'\frac{\partial f}{\partial z}=0$$

be two pairs of bitangents of the quartic curve, so that (x_1', y_1', z_1'), (x_2', y_2', z_2') are two intersections of the Hessian H and the conic $x'z'-y'^2=0$, and $(x_1'+\lambda x_2',\ y_1'+\lambda y_2',\ z_1'+\lambda z_2')$ be the third intersection of their join with the Hessian H, then the conic

$$(x_1'+\lambda x_2')\frac{\partial f}{\partial x}+\ldots=0$$

is also a pair of straight lines which, though not bitangents of the quartic curve, also touch the Cayleyan Γ. Thus the twelve bitangents of a Steiner system consist of six tangents of the Cayleyan which can be denoted, say by u_1, u_2, u_3, u_4, u_5, u_6, and six lines corresponding to these respectively which can be denoted by $u_1+\omega$, $u_2+\omega$, $u_3+\omega$, $u_4+\omega$, $u_5+\omega$, $u_6+\omega$, where ω is a half-period for the elliptic functions in terms of which the tangents of the Cayleyan Γ can be expressed.

To examine this further we may find the relation connecting the elliptic representation of the Hessian H, which is a cubic curve, and the class-cubic Γ. If the tangent of the latter be written

$$x(\wp u-e)+y\wp' u=1,$$

the periods of the functions being 2ω, $2\omega'$, it is at once proved that the point of intersection of the tangents u, $u+\omega$ is given by

$$2mx=\wp_1(u)-e, \qquad 2my=-\tfrac{1}{4}\wp_1'(u)/[\wp_1(u)+2e],$$

where

$$\wp_1(u)=\wp(u)+\wp(u+\omega)-e$$

is the $\wp$-function of periods ω, $2\omega'$ which satisfies the differential equation

$$\wp_1'^2 = 4\wp_1^3 - g_2'\wp_1 - g_3',$$

wherein $g_2' = 4(4m+3e^2)$, $g_3' = 8e(4m-e^2)$, $m = (e-e')(e-e'')$;

this is equivalent to the statement that

$$[\wp(u)-e][\wp_1(u)-e] - \tfrac{1}{4}\frac{\wp'(u)\,\wp_1'(u)}{\wp_1(u)+2e} = 2m,$$

which is easily verified. The functions

$$\wp_1(u)-e, \quad -\tfrac{1}{4}\wp_1'(u)/[\wp_1(u)+2e]$$

have for poles, in the parallelogram $(\omega, 2\omega')$, respectively $u = 0$ twice and $u = 0$, $u = \omega'$ each once; thus, if three points of the Hessian of which these functions give the elliptic expression lie upon a straight line, the sum of the corresponding arguments u is $N\omega+2N'\omega'+\omega'$, where N, N' are integers, while if six points of the Hessian lie upon a conic, their arguments have a sum of the form $M\omega+2M'\omega'$, where M, M' are integers. Thus we can suppose the twelve tangents of the Cayleyan which form a Steiner system to be associated with arguments

$$\begin{array}{cccccc} u_1, & u_2, & u_3, & u_4+\omega, & u_5+\omega, & u_6+\omega, \\ u_1+\omega, & u_2+\omega, & u_3+\omega, & u_4, & u_5, & u_6, \end{array}$$

wherein $$u_1+u_2+u_3+u_4+u_5+u_6 \equiv 0 \pmod{2\omega, 2\omega'};$$

from this it appears that any odd number of elements from the first row taken with the elements of different columns in the second row give rise to six lines chosen from the twelve which are all tangents of the same conic (cf. the last example of § 11).

Another set of twelve tangents of the Cayleyan which are similarly touched in sixes by conics is given by

$$\begin{array}{cccccc} u_1, & u_2, & u_3, & u_4+\omega', & u_5+\omega', & u_6+\omega', \\ u_1+\omega', & u_2+\omega', & u_3+\omega', & u_4, & u_5, & u_6, \end{array}$$

where ω' is another half period than ω; from these together can be formed another set

$$\begin{array}{cccccc} u_1+\omega, & u_2+\omega, & u_3+\omega, & u_4+\omega', & u_5+\omega', & u_6+\omega', \\ u_1+\omega', & u_2+\omega', & u_3+\omega', & u_4+\omega, & u_5+\omega, & u_6+\omega. \end{array}$$

(Cf. § II of the Appendix below.)

It is also to be remarked that the points of intersection of the pairs of a Steiner system of bitangents of a plane quartic curve may be regarded

as the six poles of their plane, in regard to six linear complexes of which every two are in involution, the quartic curve itself being a section of a quartic surface with sixteen nodes. See Ciani, *Ann. di Mat.*, t. II (1899), p. 93, and the author's "Multiply-Periodic Functions," pp. 319, 320 foot-note.

5. *The Quartic Curve as derived from a Cubic Surface.*

If, as before, ψ, ϕ, χ be homogeneous quadric functions in x, y, z, the quartic surface

$$x^2\psi+2xt\phi+t^2\chi=0,$$

wherein x, y, z, t are homogeneous space coordinates, has a node at $x=y=z=0$, and the tangent lines of the surface from this node touch the quartic cone

$$\psi\chi=\phi^2;$$

the line $x=0$, $t=0$ is a double line of the surface, and the plane $t=\mu x$ cuts the surface again in a curve lying on the quadric cone

$$\psi+2\mu\phi+\mu^2\chi=0,$$

whose vertex is $x=y=z=0$; this quadric cone touches the quartic cone whose vertex is the node along four generators, as we have previously seen. Taking ξ to be any homogeneous linear function of x, y, z, the transformation from t to T,

$$t=x\,\frac{\mu_1\xi+\mu_2 T}{\xi+T},$$

gives to the quartic surface the form

$$\xi^2(\psi+2\mu_1\phi+\mu_1^2\chi)+2\xi T[\psi+(\mu_1+\mu_2)\phi+\mu_1\mu_2\chi]+T^2(\psi+2\mu_2\phi+\mu_2^2\chi)=0.$$

We may suppose μ_2 to be such that $\psi+2\mu_2\phi+\mu_2^2\chi$ breaks into two linear factors, say is equal to ξP, taking ξ above to be the form thus determined, and, as a particular case, may suppose $\mu_1=0$; then the quartic surface becomes, omitting the factor ξ, the cubic surface

$$\xi\psi+2T(\psi+\mu_2\phi)+T^2P=0,$$

wherein we may suppose ψ, ϕ, P expressed homogeneously in terms of ξ, y, and z.

Conversely, taking any cubic surface and choosing the coordinates so that $x=0=t$ represents a line lying upon the surface, and $x=y=z=0$ is a point upon the surface, the equation is of the form

$$x\psi+2t\phi+t^2P=0,$$

where ψ, ϕ are quadric cones with vertex at $x = y = z = 0$, and $P = 0$ is the tangent plane at this point. The tangent lines of this surface drawn from $x = y = z = 0$ lie upon the quartic cone $\phi^2 = \psi x P$, which, as we have seen, is a perfectly general quartic; the curve of contact of this cone, which lies also upon the quadric $\phi + tP = 0$, is a sextic curve with a double point at $x = y = z = 0$; the quadric $\phi + tP = 0$ is the polar quadric of $x = y = z = 0$ in regard to the cubic surface. Any plane $t = \mu x$ cuts the cubic surface upon the quadric cone

$$\psi + 2\mu\phi + \mu^2 x P = 0,$$

of which, as we have seen, four tangent planes are tangent planes of the cubic surface; the plane $t = \mu x$ meets the sextic curve upon four generators of contact, and in the points where the cone

$$\psi + 2\mu\phi + \mu^2 x P = 0 \text{ cuts } x = 0,\ t = 0.$$

There are five values of μ for which $\psi + 2\mu\phi + \mu^2 x P$ breaks up into two planes, or five pairs of lines upon the cubic surface lying in five planes through any arbitrary line upon the surface.

From this, and the fact that a plane containing two lines of the surface must contain a third, it is easy to see that, starting from an arbitrary line $B''C''$ of the surface, and a pair of lines $P'BC$, $PB'C'$ of the surface intersecting this, we can, in various ways, construct three triangles of lines upon the surface $A''B''C''$, $A'B'C'$, ABC, such that $AA'A''$, $BB'B''$, $CC'C''$ are three straight lines, not upon the surface, meeting in one point, and also the three triangles of lines of the surface formed respectively by BC, $B'C'$, $B''C''$; CA, $C'A'$, $C''A''$; AB, $A'B'$, $A''B''$ are in perspective. If the planes of ABC, $A'B'C'$, $A''B''C''$ be called respectively $U = 0$, $V = 0$, $W = 0$, and the planes of BC, $B'C'$, $B''C''$, ..., be called respectively $U' = 0$, $V' = 0$, $W' = 0$, the equation of the cubic surface is thence of the form

$$UVW = \lambda U'V'W'.$$

The same result can be obtained at once analytically. The equation

$$x\psi + 2t\phi + t^2 P = 0,$$

is the same as $$xy_1z_1 + 2t_1\phi_1 + t_1^2 P = 0,$$

where $t_1 = t - \mu x$, and μ is such that

$$\psi + 2\mu\phi + \mu^2 x P = y_1 z_1,$$

while $$\phi_1 = \phi + \mu x P;$$

now choosing σ so that $y_1 z_1 - 2\sigma\phi_1 + \sigma^2 x P = vw$,

and putting $x_1 = \sigma x$, $P_1 = \lambda P$, the cubic surface becomes

$$x_1 y_1 z_1 + t_1(y_1 z_1 + x_1 P_1 - vw) + t_1^2 P_1 = 0;$$

this can be identified with

$$(t_1+x_1)(t_1+by_1)(t_1+cz_1) = t_1(t_1+x_1+qv)(t_1+x_1+rw);$$

for this it is found at once that the pair of constants q, r can be chosen in one of four possible ways, and that thence b, c are uniquely determinable; putting

$$y_2 = -by_1, \quad z_2 = -cz_1, \quad v_1 = qv, \quad w_1 = rw, \quad P_2 = bc\sigma P;$$

this is the same as

$$(t_1+x_1)(t_1-y_2)(t_1-z_2) = t_1(t_1+x_1+v_1)(t_1+x_1+w_1),$$

with

$$x_1+y_2+z_2+v_1+w_1+P_2 = 0.$$

6. *The Systematic Determination of the Lines upon the Cubic Surface.*

Consider now in general a cubic surface

$$UVW = U'V'W',$$

and suppose $x = y = z = 0$ to be upon the surface, so that we may suppose the equation to be

$$(t+u)(t+v)(t+w) = (t+u')(t+v')(t+w'),$$

where $u, v, \ldots, w'$ are homogeneous in x, y, z, and $U = t+u, \ldots$. Then, if we write the two necessary linear relations connecting $U, V, \ldots, W'$ in the forms

$$A_1 U + B_1 V + C_1 W = A_1' U' + B_1' V' + C_1' W',$$

$$A_2 U + B_2 V + C_2 W = A_2' U' + B_2' V' + C_2' W',$$

we shall have

$$A_1+B_1+C_1 = A_1'+B_1'+C_1',$$

$$A_2+B_2+C_2 = A_2'+B_2'+C_2',$$

and hence, for arbitrary σ,

$$(A_1+\sigma A_2)+(B_1+\sigma B_2)+(C_1+\sigma C_2) = (A_1'+\sigma A_2')+(B_1'+\sigma B_2')+(C_1'+\sigma C_2');$$

we can, however, suppose at the same time, by choosing σ in any one of three ways, that

$$(A_1+\sigma A_2)(B_1+\sigma B_2)(C_1+\sigma C_2) = (A_1'+\sigma A_2')(B_1'+\sigma B_2')(C_1'+\sigma C_2').$$

Now we remark that, if we have six quantities $\lambda, \mu, \nu, \lambda', \mu', \nu'$ subject to

$$\lambda+\mu+\nu = \lambda'+\mu'+\nu', \quad \lambda\mu\nu = \lambda'\mu'\nu';$$

these lead to

$$\frac{(\lambda-\mu')(\lambda-\nu')}{\lambda(\mu-\mu')(\nu-\nu')} = \frac{\mu'\nu'+\nu'\lambda'+\lambda'\mu'-(\mu\nu+\nu\lambda+\lambda\mu)}{(\lambda-\lambda')(\mu-\mu')(\nu-\nu')} = \theta, \text{ say};$$

thus if we put $a = \dfrac{\mu'-\nu}{\lambda-\lambda'}$, $b = \dfrac{\nu'-\lambda}{\mu-\mu'}$, $c = \dfrac{\lambda'-\mu}{\nu-\nu'}$,

we obtain $\theta\lambda = b(1-c)$, $\theta\mu = c(1-a)$, $\theta\nu = a(1-b)$,

$$\theta\lambda' = (1-b)c, \quad \theta\mu' = (1-c)a, \quad \theta\nu' = (1-a)b,$$

or λ, μ, ν, λ', μ', ν' are in the ratios of bc', ca', ab', $b'c$, $c'a$, $a'b$, where

$$a' = 1-a, \quad b' = 1-b, \quad c' = 1-c.$$

We are thus entitled to put the identities connecting $U, V, \ldots, W'$ in the three forms

$$b_i c_i' U + c_i a_i' V + a_i b_i' W = b_i' c_i U' + c_i' a_i V' + a_i' b_i W',$$

for $i = 1, 2, 3$, the three relations being equivalent to two. Then putting

$$\xi_i = b_i c_i' U, \quad \eta_i = c_i a_i' V, \quad \zeta_i = a_i b_i' W,$$

$$\xi_i' = b_i' c_i U', \quad \eta_i' = c_i' a_i V', \quad \zeta_i' = a_i' b_i W',$$

the cubic surface has the form, which is one of three related as here,

$$\xi_i \eta_i \zeta_i = \xi_i' \eta_i' \zeta_i',$$

wherein $\xi_i, \ldots, \xi_i'$ are subject to two linear relations of which one is

$$\xi_i + \eta_i + \zeta_i = \xi_i' + \eta_i' + \zeta_i'.$$

The two relations $\xi\eta\zeta = \xi'\eta'\zeta'$, $\xi+\eta+\zeta = \xi'+\eta'+\zeta'$ alone may be regarded as representing a variety (threefold) in four dimensions; this is that known as Segre's cubic variety; the cubic surface is thence a section of this, by an arbitrary planar threefold.

With the forms obtained for the cubic surface we can at once write down the twenty-seven lines which lie upon it. There are, first, the nine lines

$$\begin{array}{lll} \xi\xi' & \xi\eta' & \xi\zeta' \\ \eta\xi' & \eta\eta' & \eta\zeta' \\ \zeta\xi' & \zeta\eta' & \zeta\zeta', \end{array}$$

wherein, for example, $\xi\xi'$ denotes the line given by $\xi = 0$, $\xi' = 0$, that

is $U = 0$, $U' = 0$, and it is immaterial to supply the suffix; and then there are the eighteen lines which we shall denote by

$$(\xi'\eta'\zeta')_i, \quad (\xi'\zeta'\eta')_i, \quad (\eta'\zeta'\xi')_i, \quad (\eta'\xi'\zeta')_i, \quad (\zeta'\xi'\eta')_i, \quad (\zeta'\eta'\xi')_i,$$

where i is to have each of the three values 1, 2, 3, and, for example, $(\xi'\eta'\zeta')_i$ denotes the intersection of the three planes

$$\xi_i = \xi_i', \quad \eta_i = \eta_i, \quad \zeta_i = \zeta_i',$$

or again, $(\eta'\zeta'\xi')_i$ denotes the intersection of the three planes

$$\xi_i = \eta_i', \quad \eta_i = \zeta_i', \quad \zeta_i = \xi_i'.$$

In regard to the mutual intersections of these twenty-seven lines, if we form the determinant

$$\begin{vmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{vmatrix},$$

and denote by the elements of this the lines written above as a square of nine elements, so that, for example, a_1, a_2 denote respectively the lines $(\xi\xi')$, $(\xi\eta')$, then every term in the expansion of this determinant may be regarded as denoting three lines, of different *denomination*, as we shall say, according to the value of the suffix i; for example, the term $a_1b_3c_2$ of the expansion may be regarded as denoting the lines $(\xi'\zeta'\eta')_i$. Two terms of the expansion of the determinant being called *like* when they have a common factor, so that, for example, $a_1b_3c_2$, $a_3b_2c_1$, $a_2b_1c_3$ are all like the term $a_1b_2c_3$, and *unlike* when this is not so, so that, for example, $a_2b_3c_1$ and $a_3b_1c_2$ are both unlike $a_1b_2c_3$, the intersection properties of the twenty-seven lines may be summed up by saying—

(1) A line represented by any of the nine elements of the determinant intersects lines represented by elements in the same row or column of the determinant, and also intersects lines of all denominations represented by terms of the expansion having that element as factor; this gives the total number $10 = 4+3.2$ of lines intersecting the line represented by the element.

(2) A line represented by any term of the expansion intersects all lines represented by like terms in the expansion which are of the same denomination, and all lines represented by unlike terms which are of different denomination, besides intersecting lines represented by elements

of the determinant occurring as factor in the term in question; this equally gives all the $10 = 3+2.2+3$ lines intersecting the line under consideration.

These statements are easy to verify; they shew too that the surface contains only the twenty-seven lines previously specified. It is easy also to verify such statements as that two non-intersecting lines of the surface are intersected by five other non-intersecting lines of the surface, and that three non-intersecting lines of the surface are met by three other non-intersecting lines.

There is a different way of formulating, by a single rule, the intersections of the lines. This is by first giving a name, according to the following plan, to each of the lines. Consider eight symbols u, v, a, b, c, f, g, h, which we may consider as capable of algebraic multiplication, obeying the commutative and associative laws, assuming then that the square of each symbol is unity and that the total product $uvabcfgh$ is also unity; with the seven lines of the surface

$$\xi\xi', \quad \xi\eta', \quad \eta\xi', \quad \eta\eta', \quad (\xi'\eta'\zeta')_1, \quad (\xi'\eta'\zeta')_2, \quad (\xi'\eta'\zeta')_3,$$

associate respectively the binary products

$$ab, \quad au, \quad av, \quad ac, \quad gh, \quad hf, \quad fg;$$

any two intersecting lines of the surface have a third line of the surface lying in their plane; applying this fact to pairs of intersecting lines among the seven given lines we obtain others; using the original seven and these others, the same process gives still others, and so on; and it is easy to see that so all the possible twenty-seven lines are obtainable. From the eight symbols $u, v, \ldots, h$ we can form twenty-eight products of two; when one of these twenty-eight binary products has been associated with each of two intersecting lines of the surface, neither of these being uv, we can agree to associate with the third line in the plane of these two such a symbol that the product of the symbols of the three coplanar lines shall be uv; for instance, the lines $\xi\xi'$, $\xi\eta'$ being respectively ab, au, the third line in their plane $\xi\zeta'$, will thus obtain the symbol bv. It is easily seen on trial that by this rule every line found in this way from the primary seven obtains a binary symbol, the only one of the twenty-eight binary symbols that are possible which is not utilised being uv. But the question arises whether—as at any stage of the process of obtaining the lines it may be possible to obtain a new line in different ways from pairs

of intersecting lines already obtained—the final symbol obtained for any line is without ambiguity. That this is so is clear on enumeration: on the one hand from the fact that every line of the surface is a side of ten triangles of lines, it follows that there are forty-five triangles of lines; on the other hand, every triad of binary symbols having the product uv must either be a triad ua, vb, ab, where a, b denote any two of the six a, b, c, f, g, h, or else a triad ab, cf, gh, formed by dividing these six symbols into three pairs; there are thus forty-five triads of binary symbols, easily enumerated and shewn to correspond each to an actually existing triangle of lines. The final result is given by the table

$$\left.\begin{matrix} \xi\xi' & \xi\eta' & \xi\zeta' \\ \eta\xi' & \eta\eta' & \eta\zeta' \\ \zeta\xi' & \zeta\eta' & \zeta\zeta' \end{matrix}\right\} \equiv \left\{\begin{matrix} ab & au & bv \\ av & ca & cu \\ bu & cv & bc \end{matrix}\right.$$

$(\xi'\eta'\zeta')_i$	$(\xi'\zeta'\eta')_i$	$(\eta'\zeta'\xi')_i$	$(\eta'\xi'\zeta')_i$	$(\zeta'\xi'\eta')_i$	$(\zeta'\eta'\xi')_i$
gh, hf, fg	cf, cg, ch	vf, vg, vh	af, ag, ah	uf, ug, uh	$bf, bg, bh,$

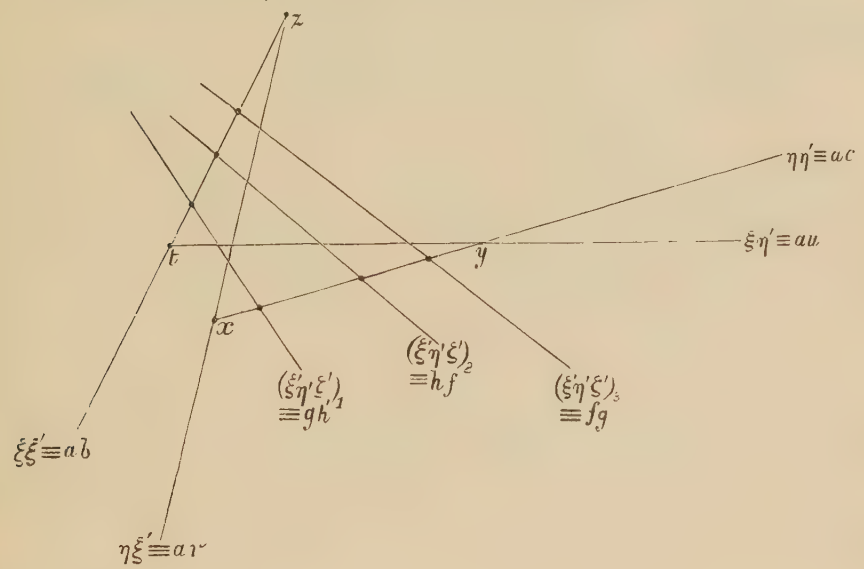
where, in the first part, similarly placed elements of the squares denote the same lines, and, in the second part, the suffix i is in turn to have the values 1, 2, 3, and the resulting lines are those associated in order with the underlying symbols; it will be seen that the various values of i always correspond to a circulation of the letters f, g, h; and that, if X', Y', Z' denote ξ', η', ζ' in some order, and α, β, γ be the respective symbols of the lines $\xi X', \eta Y', \zeta Z'$, the symbols for the lines $(X'Y'Z')_i$ are respectively $\alpha\beta\gamma gh, \alpha\beta\gamma hf, \alpha\beta\gamma fg$.

These symbols are essentially those, arising from the theory of Theta-functions, for the bitangents of a plane quartic curve; we do not therefore examine in what ways those arbitrarily assigned to the seven primary lines may be changed, nor what are the possible sets of seven lines of the cubic surface which may be taken as primary. Information in regard to the former of these questions may be found in the author's *Abel's Theorem*, Chap. XVII, p. 505. But it will be of interest to examine the geometrical figure formed by the seven primary lines of the surface here taken.

7. *Determination of the Lines of the Cubic Surface from Seven Primary Lines.*

These seven primary lines, in fact, form a skew quadrilateral together with three non-intersecting lines, meeting alternate sides of the quadri-

lateral; or, they consist of two non-intersecting lines and the five non-intersecting lines which meet these. This is indicated by the figure



Taking the angular points of the quadrilateral as corners of a tetrahedron of reference $xyzt$, the general cubic surface containing the quadrilateral is

$$xt(ax+by+cz+dt)-yz(a'x+b'y+c'z+d't)=0;$$

the conditions that this surface should contain the line $y=mx$, $z=nt$ are

$$a+bm+mn(a'+b'm)=0, \quad nc+d+mn(nc'+d')=0,$$

and three such lines being given the surface is of the form

$$xt[\lambda(x+Ay)+\mu(t+Bz)]+yz[\lambda(Cx+Dy)+\mu(Et+Fz)]=0,$$

wherein λ, μ are arbitrary. With the seven given lines the surface is then determined when one further point is given; as is obvious also by counting the number of conditions imposed by the given lines. Putting

$$U=x, \quad V=t, \quad U'=y, \quad V'=z,$$

$$W=ax+by+cz+dt, \quad W'=a'x+b'y+c'z+d't,$$

the surface is $$UVW=U'V'W',$$

with
$$aU+dV-W+bU'+cV' \qquad =0,$$
$$a'U+d'V \qquad +b'U'+c'V'-W'=0;$$

thus (§ 6), if σ_1, σ_2, σ_3 be the roots of the cubic equation

$$(a+\sigma a')(d+\sigma d')(-1) = (b+\sigma b')(c+\sigma c')\sigma,$$

by putting

$$\xi_i = (a+\sigma_i a')\,U, \qquad \eta_i = (d+\sigma_i d')\,V, \qquad \zeta_i = -\,W,$$
$$\xi_i' = -(b+\sigma_i b')\,U', \quad \eta_i' = -(c+\sigma_i c')\,V', \quad \zeta_i' = \sigma_i W',$$

the surface is reduced to the form which has been discussed, and all the twenty-seven lines are determinable rationally in terms of the coefficients a, b, ..., d' and σ_1, σ_2, σ_3; in particular, the three lines $(\xi'\eta'\zeta')_i$ are

$$(a+\sigma_i a')x = -(b+\sigma_i b')y, \quad (d+\sigma_i d')t = -(c+\sigma_i c')z;$$

as all these are given, so also are σ_1, σ_2, σ_3.

It thus appears that all the lines of the surface are rationally determinable when the seven primary lines are given.

It will be seen below (§ 11) that the figure gives an immediate proof of Aronhold's determination of the plane quartic curve from seven given bitangents.

8. *Note.—Determination of Riemann's Equations for the Bitangents of a Plane Quartic Curve.*

It may be worth while to indicate the algebra by which Riemann's equations for the bitangents of a plane quartic curve can be deduced from what precedes (Riemann, *Ges. Werke*, 1876, p. 466).

In the relation

$$bc'U+ca'V+ab'W = b'cU'+c'aV'+a'bW'$$

take

$$p = U-U', \quad q = W'-U, \quad r = U'-V,$$

$$p' = U+V+W-(U'+V'+W'), \quad q' = V'-U, \quad r' = U'-W,$$

so that p, q, r, p', q', r' represent planes passing through the same point of the cubic surface, and

$$p+q+r+p'+q'+r' = 0,$$

$$V = U-p-r, \quad W = U-p-r', \quad U' = U-p, \quad V' = U+q', \quad W' = U+q.$$

Then the relation in question, by subtraction of $aa'(p+q+r+p'+q'+r')$, becomes

$$(a-b)(a-c)\,p-a'(a-b)q+a'(c-a)r-aa'p'-a(c-a)q'+a(a-b)r' = 0.$$

Now take quantities λ, A, B, C, so that

$$\lambda^2 aa'(a-b)(a-c) = -1,$$

$$A = -\lambda(a-b)(a-c), \quad B = \lambda a'(a-b), \quad C = -\lambda b'(c-a);$$

these give rise to

$$1/A = \lambda aa', \quad 1/B = \lambda a(c-a), \quad 1/C = -\lambda a(a-b),$$

$$a = \frac{1}{1-BC}, \quad b = \frac{1-AB}{1-BC}, \quad c = \frac{1-AC}{1-BC}.$$

There will be three sets of values of A, B, C corresponding to the three sets of values of a, b, c, and the linear relations connecting p, q, r, p', q', r' are

$$p+q+r+p'+q'+r' = 0,$$

$$A_1 p+B_1 q+C_1 r+\frac{1}{A_1}p'+\frac{1}{B_1}q'+\frac{1}{C_1}r' = 0,$$

$$A_2 p+B_2 q+C_2 r+\frac{1}{A_2}p'+\frac{1}{B_2}q'+\frac{1}{C_2}r' = 0,$$

$$A_3 p+B_3 q+C_3 r+\frac{1}{A_3}p'+\frac{1}{B_3}q'+\frac{1}{C_3}r' = 0,$$

equivalent to only three independent equations.

The cubic surface has the equation

$$U(U-p-r)(U-p-r') = (U-p)(U+q')(U+q);$$

putting $U = t+\omega$, where $\omega = 0$ is a plane through the intersection of p, q, r, p', q', r'; this is

$$(t+u)(t+v)(t+w) = (t+u')(t+v')(t+w'),$$

where $u = \omega, \quad v = \omega-p-r, \quad w = \omega-p-r',$

$$u' = \omega-p, \quad v' = \omega+q', \quad w' = \omega+q;$$

the tangent cone to this from the point t is

$$(vw+wu+uv-v'w'-w'u'-u'v')^2$$

$$= 4(uvw-u'v'w')(u+v+w-u'-v'-w'),$$

which reduces to $\sqrt{pp'}+\sqrt{qq'}+\sqrt{rr'} = 0.$

Taking now the twenty-seven lines of the cubic surface, as given in § 6, substituting for $\xi, \eta, \zeta, \xi', \eta', \zeta'$ in terms of U, V, W, U', V', W' and a, b, c, a', b', c', we can find for each line, without much difficulty, a

homogeneous relation in $p,\ q,\ r,\ p',\ q',\ r'$. For instance, the line with the binary symbol cf, that is $(\xi'\zeta'\eta')$, is given by

$$\xi = \xi', \quad \eta = \zeta', \quad \zeta = \eta',$$

or

$$bc'U = b'cU', \quad ca'V = a'bW';$$

this line, therefore, lies upon the plane

$$Ubc' - (U-p)b'c + c(U-p-r) - b(U+q) = 0,$$

which reduces to

$$bcp + bq + cr = 0, \quad \text{or} \quad \frac{p}{1-BC} + \frac{q}{1-CA} + \frac{r}{1-AB} = 0.$$

Proceeding in this way, we find for the twenty-eight bitangents of the binary quartic curve

$$\sqrt{pp'} + \sqrt{qq'} + \sqrt{rr'} = 0,$$

the forms

$$\begin{array}{lll}
(ab)\ p = 0, & (bv)\ q = 0, & (av)\ r = 0, \\
(uv)\ p' = 0, & (au)\ q' = 0, & (bu)\ r' = 0, \\
(cv)\ p' + q + r = 0, & (ca)\ q' + r + p = 0, & (bc)\ r' + p + q = 0, \\
(uf)\ A_1 p + B_1 q + C_1 r = 0, & (cu)\ p + q + r = 0, &
\end{array}$$

$$(vf)\ A_1^{-1} p' + B_1 q + C_1 r = 0,$$

$$(gh)\ A_1^{-1}(1-B_1C_1)^{-1} p' + B_1^{-1}(1-C_1A_1)^{-1} q' + C_1^{-1}(1-A_1B_1)^{-1} r' = 0,$$

$$(af)\ B_1^{-1} q' + C_1 r + A_1 p = 0,$$

$$(cf)\ (1-B_1C_1)^{-1} p + (1-C_1A_1)^{-1} q + (1-A_1B_1)^{-1} r = 0,$$

$$(bf)\ C_1^{-1} r' + A_1 p + B_1 q = 0,$$

where in each case we have prefixed the binary symbol associated with the bitangent, and a change of the symbol f into g is to be accompanied by a change of $A_1,\ B_1,\ C_1$ into $A_2,\ B_2,\ C_2$, a change of f into h by a change of $A_1,\ B_1,\ C_1$ into $A_3,\ B_3,\ C_3$.

9. *The Correspondence between the Lines of the Cubic Surface and the Bitangents of the Plane Quartic Curve.*

We have seen that the tangent cone drawn to a cubic surface, from any point O of itself, gives, by taking an arbitrary plane section of it, a general quartic curve, and that conversely a cubic surface can in various ways be found to correspond to a given plane quartic. The plane joining

O to any line of the surface will contain, beside the line, also a conic upon the surface, the two points of intersection of the line and conic are points of contact of the plane and the cubic surface (and are also points of contact of the plane with the sextic curve in which the cone of contact touches the cubic surface); thus the line projects into a bitangent of the quartic curve. Similarly, the sextic curve of contact lying upon the polar quadric of O in regard to the cubic surface, the tangent plane of the cubic surface at O gives rise to a bitangent of the quartic curve whose points of contact are on the generators of the quadric at O, these being the tangents at the double point of the sextic at O.

Take now three lines of the cubic surface which lie in one plane; arbitrary planes through these lines each intersect the cubic surface also in a conic; every two of these, as is easy to see, have two points in common; and three conics in space, of which every two have two points in common, lie upon a quadric surface. Thus, in particular, taking the three planes to be concurrent in O, we infer that the three coplanar lines of the cubic surface project into three bitangents of the plane quartic whose six points of contact are on a conic. Analytically, the cubic surface being $xyz+2t\phi+t^2P=0$, where $x=0$, $y=0$, $z=0$ is the point O, and $t=0$ is the plane of the three lines of the cubic surface, the conics lying in the three planes $t=\lambda x$, $t=\mu y$, $t=\mu z$ lie on the quadric $Q=0$ where, identically,

$$tQ=(t-\lambda x)(t-\mu y)(t-\mu z)+\lambda\mu\nu(xyz+2t\phi+t^2P),$$

and when the planes are $x=0$, $y=0$, $z=0$, this quadric becomes $2\phi+tP=0$. This quadric, like the polar conic of O whose equation is $\phi+tP=0$, contains the tangents of the sextic curve at O. Thus the conic containing the points of contact of the bitangents of the quartic curve obtained by projection of the three coplanar lines of the cubic surface contains also the points of contact of the bitangent of the quartic curve which lies in the tangent plane of the cubic surface at O.

Take next two pairs of lines of the cubic surface intersecting a line c of the surface, say a, b' which intersect c and each other, and a', b which intersect c and each other. Then we can prove that the eight points of contact of the bitangents into which a, b', a', b project, all lie upon a conic. This is clear from the fact that, if any two conics be drawn through the four points of contact of any one of the contact conics of the quartic, of the form $\psi+2\mu\phi+\mu^2\chi=0$ (see § 1), their other intersections with the quartic, eight in all, lie upon another conic; if these two conics be

$$\psi+(\mu+\rho)\phi+\mu\rho\chi=0, \quad \psi+(\mu+\sigma)\phi+\mu\sigma\chi=0,$$

the quartic curve itself being capable of either of the forms

$$[\psi+(\mu+\rho)\phi+\mu\rho\chi]^2=[\psi+\mu\phi+\mu^2\chi][\psi+\rho\phi+\rho^2\chi],$$
$$[\psi+(\mu+\sigma)\phi+\mu\sigma\chi]^2=[\psi+\mu\phi+\mu^2\chi][\psi+\sigma\phi+\sigma^2\chi],$$
$$[\psi+(\rho+\sigma)\phi+\rho\sigma\chi]^2=[\psi+\rho\phi+\rho^2\chi][\psi+\sigma\phi+\sigma^2\chi],$$

the eight points of intersection in question lie upon the conic

$$\psi+(\rho+\sigma)\phi+\rho\sigma\chi=0;$$

denoting by ω the bitangent of the plane quartic which lies on the tangent plane of the cubic surface at O, the bitangents a, b', c, ω have their points of contact upon a conic, as we have seen, as have also the bitangents a', b, c, ω; hence, by the result enunciated, so have the bitangents a, b', a', b, as was to be proved.

Now a Steiner system of bitangents of the plane quartic is obtainable, by definition, by taking two arbitrary bitangents, and adding thereto all other bitangents of which the two points of contact lie on a conic through the four points of contact of the two arbitrarily selected bitangents; the same conic always contains the four points of contact of two of the ten bitangents so determined from the first two; there are thus

$$\frac{1}{6}\cdot\frac{28.27}{2}=63$$

Steiner systems, and of these there are twenty-seven systems in which one pair of bitangents contains a specified bitangent ω as one of its components. Consider a particular cubic surface of which the cone of contact from a point O of itself gives rise to the quartic curve, and let ω be the bitangent lying in the tangent plane at O; we have seen that three coplanar lines of the surface, say p, q, r, give bitangents whose points of contact lie with those of ω upon a conic. Hence r, ω define for the quartic curve a Steiner system containing p, q as another pair, and therefore containing as pairs the projections of each of the five pairs of coplanar lines of the surface which intersect r. The twenty-seven Steiner systems of the plane quartic curve which contain ω are thus obtainable from the cubic surface by pairing ω with each of the twenty-seven lines of the surface in turn. Again, taking any one of the twenty-seven lines of the cubic surface, there are ten lines of the surface which intersect it, and therefore sixteen which do not intersect it; hence there are $\frac{1}{2}\,27.16=6.36$ pairs of non-intersecting lines on the surface. Taking such a pair a, b, the line b must meet one of each of the five pairs of intersecting lines which meet a; say the five non-intersecting lines which meet a, b are c_1, c_2, ..., c_5; then the plane ac_i determines a line b_i meeting a, and the

plane bc_i a line a_i meeting b; we have shewn above that the four lines ab_ia_ib project into bitangents forming two pairs of a Steiner system, which we may suppose determined by a, b; we obtain thus a complete Steiner system by taking $i = 1, \ldots, 5$. As each of these contains six pairs of lines, and the lines a_i, b_i do not intersect, we can thus obtain $\frac{1}{6}.6.36 = 36$ Steiner systems. These, with the twenty-seven previously explained, containing ω as one bitangent, make a tale of sixty-three, the total number. It thus appears that every Steiner system is obtainable in one of these two ways.

10. *The Correspondence in connexion with the Notation for the Lines. Double Sixes.*

There is advantage in placing these results in connexion with the notation previously explained. From what has been said it appears that each of the twenty-eight bitangents of a plane quartic can be represented by one of the binary combinations of eight symbols a, b, c, f, g, h, u, v; and that the pairs of a Steiner system of bitangents are determined by the condition that the product of the binary symbols attached to the bitangents of a pair is the same for each of the six pairs. This product is itself necessarily either a binary symbol or a quaternary symbol, and it will be found on trial that either a binary or a quaternary symbol can be written as the product of two binary symbols in six ways only. For instance, a binary symbol uv can be written as the product of the symbols in the six following columns

$$\left.\begin{array}{cccccc} au & bu & cu & fu & gu & hu \\ av & bv & cv & fv & gv & hv \end{array}\right\}, \qquad \text{(I)}$$

and there are twenty-eight Steiner systems of this form; while a quaternary symbol $uabc$, which is the same as $vfgh$, can be written as the product of the elements in the columns

$$\left.\begin{array}{cccccc} ua & ub & uc & gh & hf & fg \\ bc & ca & ab & vf & vg & vh \end{array}\right\} \qquad \text{(II)}$$

and there are $\frac{1}{2}\,8!/4!\,4! = 35$ Steiner systems of this form.

In this enumeration no distinction is made between u, v and other of the eight letters; on the cubic surface, however, the eight letters do not occur in a symmetrical way. A Steiner system may arise either from

eleven lines of the cubic surface of the form

$$\left.\begin{matrix} ab & au & av & gh & hf & fg \\ & bv & bu & cf & cg & ch \end{matrix}\right\} (\alpha),$$

taken with the line uv lying in the tangent plane of the cubic surface at the point of projection, and there are twenty-seven possibilities of this form, which we may speak of as twenty-seven improper Schläfli systems on the cubic surface; or a Steiner system may arise from the single type

$$\left.\begin{matrix} au & bu & cu & fu & gu & hu \\ av & bv & cv & fv & gv & hv \end{matrix}\right\} (\beta),$$

or from the type

$$\left.\begin{matrix} au & av & bc & bf & bg & bh \\ bu & bv & ac & af & ag & ah \end{matrix}\right\} (\gamma),$$

of which there are fifteen possibilities, or from the type

$$\left.\begin{matrix} ua & ub & uc & gh & hf & fg \\ bc & ca & ab & vf & vg & vh \end{matrix}\right\} (\delta),$$

of which there are twenty possibilities; the $1+15+20$ possibilities included in the types (β), (γ), (δ) may be denoted as the thirty-six proper Schläfli systems on the cubic surface. These are those known as double sixes; denoting the lines by

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6, \end{matrix}$$

it is at once found that any a_i intersects every b_j except b_i, and also that a_i, b_j determine, as the third line in their plane, the same line c_{ij} of the cubic surface as do a_j, b_i; and further that the fifteen lines c_{ij} so arising are the other lines of the cubic surface beside those of the double six. A line c_{ij} intersects the four lines a_i, a_j, b_i, b_j, and also every one of the six lines c_{rs} in which neither r nor s is equal to i or j. The thirty-six systems are thus similar in configuration notwithstanding the difference of notation. Remarks in regard to the construction of such a configuration independently of the cubic surface are given below in the Appendix.

11. *Examples of the Correspondence between Steiner Systems of Bitangents and Schläfli Systems of Lines on the Cubic Surface.*

It has been seen that the cubic surface whose lines project into the bitangents of a given plane quartic is not unique; it appears sufficiently

from what precedes than an improper Schläfli system of one surface may by projection be put into relation with a proper Schläfli system of another surface.

From the fact that the six intersections of the pairs of a Steiner system of bitangents of a plane quartic curve lie upon a conic, as has been proved (§ 2), it follows that if the six transversals be drawn from any point of a cubic surface, one to meet every pair, a_i, b_i, of the six pairs of non-intersecting lines of a proper Schläfli system, or double six, these six transversals lie upon a quadric cone.

But it also follows in a similar way that if the five pairs of coplanar lines of a cubic surface be taken which meet a given line l of the surface, the quadric cone containing the five lines joining an arbitrary point O of the surface to the five intersections of these pairs, contains also the line joining O to the point of intersection of the line l and the tangent plane at O. It may be of interest to give a direct elementary verification of this result. The cubic surface being

$$xyz+t(ax^2+by^2+cz^2+2fyz+2gzx+2hxy)+t^2(ux+vy+wz)=0,$$

the plane $t=\lambda x$, through the line l, which we take as $t=0$, $x=0$, will cut the surface in two straight lines if

$$\begin{vmatrix} u\lambda^2+\lambda a, & \frac{1}{2}v\lambda^2+\lambda h, & \frac{1}{2}w\lambda^2+\lambda g \\ \frac{1}{2}v\lambda^2+\lambda h, & \lambda b, & \lambda f+\frac{1}{2} \\ \frac{1}{2}w\lambda^2+\lambda g, & \lambda f+\frac{1}{2}, & \lambda c \end{vmatrix} = 0;$$

denoting the five roots of this by $\lambda_1, \ldots, \lambda_5$, and the minors of the determinant when λ_i is written for λ by A_i, B_i, ..., the condition to be investigated is at once seen to be

$$\begin{vmatrix} A_1, & B_1, & C_1, & F_1, & G_1, & H_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ A_5, & B_5, & C_5, & F_5, & G_5, & H_5 \\ 0, & w^2, & v^2, & -wv, & 0, & 0 \end{vmatrix} = 0;$$

regarded as a polynomial in λ_1, this last determinant vanishes for the four values $\lambda_1=\lambda_2, \ldots, \lambda_1=\lambda_5$; but as we see by adding $\frac{1}{4}\lambda_1^4$ times the last row to the first row, the determinant is only of degree 3 in λ_1. It thus vanishes identically in regard to λ_1, and so also in regard to the other values of λ.

Another application of the principle of projecting from the cubic surface to the plane quartic curve is the intuitive proof of Aronhold's

theorem as to the construction of the quartic curve when seven arbitrary bitangents are given, in such a way that a conic does not pass through the six points of contact of any three of the bitangents. We have seen above (§ 7) that the twenty-seven lines of the cubic surface can be named from seven lines consisting of two non-intersecting lines and the five non-intersecting lines which meet these, and that, with an additional arbitrary point of space, these seven lines determine the cubic surface and the remaining twenty lines, by rational equations. Conversely, suppose we are given seven arbitrary lines in a plane; taking an arbitrary point O of space, joining this to the intersections of the seven coplanar lines, and taking upon the joining lines suitable points, as is possible in an infinite number of ways, we can determine a figure in space consisting, as in § 7, of a skew quadrilateral and three non-intersecting lines meeting a pair of opposite sides of this. If we draw the cubic surface passing through O and the seven lines in space now determined, the tangent cone of this cubic surface drawn from O will give upon the plane of the original seven coplanar lines a quartic curve having these as bitangents, every three of these being asyzygetic (their six points of contact not being upon a conic). There is much arbitrariness in the construction of the cubic surface, and it would be an interesting matter to investigate the relations of the surfaces which are possible.

Another illustration of the same principle may be referred to. In the scheme of a double six of lines above (§ 10), taking any three lines from the first row, say a_1, a_2, a_3, each of these meets all the lines b_4, b_5, b_6 lying in the other columns of the second row; hence the planes joining these lines to an arbitrary point, not necessarily on the cubic surface, touch a quadric cone. We thus see that there are six lines of a Steiner system of bitangents of a plane quartic which touch a conic. From this, assuming the symmetry of the lines of a Steiner system, or by a proper projection, we have the theorem for a cubic surface, that if five non-intersecting lines of a cubic surface be met by only one line of the surface, say l, then the six planes joining an arbitrary point O of the cubic surface to the five lines, and to l, touch a quadric cone. This was pointed out to me by Miss M. Long, of Girton College, and the theorems given in § 4 as to the sets of six lines of a Steiner system which touch a conic arose in generalisation of this result.

12. *On the Expression of the Equation of the Cubic Surface by a Sum of Cubes.*

Cremona has remarked (*Math. Ann.*, Bd. XIII, 1878) that by putting

$$\xi = x_2+x_3, \qquad \eta = x_3+x_1, \qquad \zeta = x_1+x_2,$$
$$\xi' = -(x_5+x_6), \quad \eta' = -(x_6+x_4), \quad \zeta' = -(x_4+x_5),$$

which lead to $2x_1 = \eta+\zeta-\xi, \ \ldots, \quad -2x_4 = \eta'+\zeta'-\xi', \ \ldots,$

and using the identity

$$(x+y+z)^3 = x^3+y^3+z^3+3(y+z)(z+x)(x+y),$$

the form to which the cubic surface is reducible, as we have seen, namely,

$$\xi\eta\zeta = \xi'\eta'\zeta' \quad \text{with} \quad \xi+\eta+\zeta = \xi'+\eta'+\zeta',$$

becomes changed to $\quad x_1^3+x_2^3+x_3^3+x_4^3+x_5^3+x_6^3 = 0,$

with $\quad x_1+x_2+x_3+x_4+x_5+x_6 = 0,$

together, of course, with another identity

$$h_1x_1+h_2x_2+h_3x_3+h_4x_4+h_5x_5+h_6x_6 = 0.$$

There are thirty-six ways of doing this, corresponding to the thirty-six double sixes, for the planes $\xi, \ldots,$ and hence the planes $x_1, \ldots,$ may be identified from the fact that the following lines form a double six

$$(\xi'\eta'\zeta')_2, \quad (\eta'\zeta'\xi')_2, \quad (\zeta'\xi'\eta')_2, \quad (\xi'\zeta'\eta')_3, \quad (\eta'\xi'\zeta')_3, \quad (\zeta'\eta'\xi')_3,$$
$$(\xi'\eta'\zeta')_3, \quad (\eta'\zeta'\xi')_3, \quad (\zeta'\xi'\eta')_3, \quad (\xi'\zeta'\eta')_2, \quad (\eta'\xi'\zeta')_2, \quad (\zeta'\eta'\xi')_2.$$

There is, however, an infinite number of ways of reducing the cubic surface to the sum of six cubes, say to the form

$$\Sigma A_r x_r^3 = 0, \quad \text{with} \quad \Sigma x_r = 0, \quad \Sigma h_r x_r = 0,$$

wherein, in general, the coefficients A_r are not equal, as in the Cremona cases. We apply here a method, originally due to Beltrami (*Rendi. Lombardo*, Jan. 2, 1879), to prove a theorem of some interest in this regard, originally obtained by Reye in a quite different manner (*Crelle*, Vol. 78, 1874). See also A. L. Dixon, *Proc. London Math. Soc.*, Ser. 2, Vol. 7, 1909, p. 389.

Assuming such an expression for the cubic surface, which may be a Cremona expression, and putting

$$\phi(\sigma) = \prod_{r=1}^{6}(\sigma-h_r), \qquad \psi(\sigma) = \prod_{s=1}^{5}(\sigma-k_s),$$

where $k_1, \ldots, k_5$ and p_6 are constants determined by the six equations

$$\psi(h_r)+p_6[\phi'(h_r)]^2 A_r^{-1} = 0,$$

and denoting by $f(\sigma)$ the expression

$$f(\sigma) = \phi(\sigma) \sum_{r=1}^{6} \frac{x_r}{\sigma - h_r},$$

so that $f(\sigma)$ is a cubic polynomial in σ for which

$$f(h_r) = x_r \phi'(h_r),$$

we have the identity $$\left[\sigma \frac{f^3(\sigma)}{\phi(\sigma)\,\psi(\sigma)}\right]_{\sigma=\infty} = 0,$$

and this gives $$\sum_{r=1}^{6} A_r x_r^3 = p_6 \sum_{s=1}^{5} B_s \xi_s^3,$$

where $$\xi_s = [\phi(k_s)]^{\frac{1}{3}} \sum_{r=1}^{6} \frac{x_r}{k_s - h_r}, \qquad B_s = [\phi(k_s)]^{\frac{1}{3}}/\psi'(k_s),$$

and $$\sum_{s=1}^{5} B_s \xi_s = 0.$$

This is Sylvester's expression of the cubic surface in terms of five cubes, which is thus at once deducible in Beltrami's manner from any expression in terms of six cubes.

Now utilising Sylvester's form, with the notation

$$ax^3+by^3+cz^3+fu^3+gv^3 = 0, \quad \text{or} \quad (ax^3) = 0,$$

subject to $$ax+by+cz+fu+gv = 0, \quad \text{or} \quad (ax) = 0,$$

the Hessian of the cubic surface is at once found to be

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}+\frac{f}{u}+\frac{g}{v} = 0, \quad \text{or} \quad \left(\frac{a}{x}\right) = 0,$$

and corresponding points (x, y, z, u, v), (x', y', z', u', v') are such that

$$xx' = yy' = zz' = uu' = vv'.$$

Take two arbitrary points of the Hessian surface, say (x_1), (x_2), so that $(a/x_1) = 0$, $(a/x_2) = 0$; let $(x_3) = (x_2) - \theta(x_1)$, and $(x_4) = (x_2) - \phi(x_1)$, be the other two points of the Hessian lying on the line joining (x_1) and (x_2), so that θ, ϕ satisfy the equation in θ,

$$\left(\frac{a}{x_2 - \theta x_1}\right) = 0;$$

let (x_1'), (x_2') be the points of the Hessian corresponding to (x_1), (x_2), and the line joining these cut the Hessian again in (x_5'), (x_6'), where

$$(x_5') = (x_1') - \rho\,(x_2'), \qquad (x_6') = (x_1') - \sigma\,(x_2');$$

thus, if (x_5), (x_6) be the points of the Hessian corresponding to (x_5'), (x_6'), we have

$$x_5 = \frac{x_1 x_2}{x_2 - \rho x_1}, \qquad x_6 = \frac{x_1 x_2}{x_2 - \sigma x_1},$$

while ρ, σ satisfy the equation in ρ,

$$\left(\frac{a x_1 x_2}{x_2 - \rho x_1}\right) = 0.$$

Then, taking (x_3'), (x_4') to correspond to (x_3), (x_4), we can at once verify that the twelve points which have been mentioned lie in sixes on six planes, and in fours on three lines, three of the six planes passing through each of the points; namely, that

the points 1, 2′, 3′, 4′, 5, 6 lie on the plane $\left(\frac{a x x_2}{x_1}\right) = 0$,

„ 2, 3′, 1′, 4′, 5, 6 „ $\left(\frac{a x x_1}{x_2}\right) = 0$,

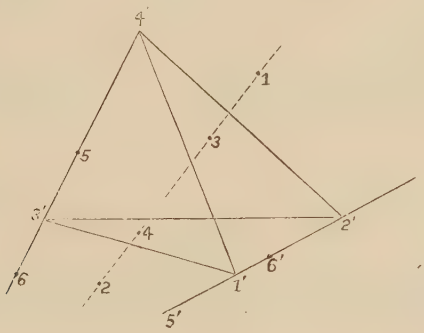
„ 3, 1′, 2′, 4′, 5, 6 „ $\left(\frac{a x x_1}{x_3}\right) = 0$,

„ 4, 1′, 2′, 3′, 5′, 6′ „ $\left(\frac{a x x_1}{x_4}\right) = 0$,

„ 1, 2, 3, 4, 5′, 6 „ $\left(\frac{a x x_1}{x_2 - \rho x_1}\right) = 0$, or $\left(\frac{a x x_5}{x_2}\right) = 0$,

„ 1, 2, 3, 4, 5, 6′ „ $\left(\frac{a x x_1}{x_2 - \sigma x_1}\right) = 0$, or $\left(\frac{a x x_6}{x_2}\right) = 0$.

This can be represented in the diagram



It will appear that the three planes passing through any one of the points form a self-conjugate trihedron in regard to the quadric cone whose vertex is at that point which is the polar quadric in regard to the cubic surface of the corresponding point of the Hessian.

Denote now by ξ, η, ζ, ω, χ the ratios $\frac{x_2}{x_1}$, $\frac{y_2}{y_1}$, $\frac{z_2}{z_1}$, $\frac{u_2}{u_1}$, $\frac{v_2}{v_1}$ of the coordinates of the point (x_2) each to the corresponding coordinate of the point (x_1); and put

$$f_1(\lambda) = (\lambda-\xi)(\lambda-\eta)(\lambda-\zeta)(\lambda-\omega)(\lambda-\chi),$$

$$f_2(\lambda) = \lambda(\lambda-\theta)(\lambda-\phi)(\lambda-\rho)(\lambda-\sigma),$$

$$f(\lambda) = f_1(\lambda)\left(\frac{ax}{\lambda-\xi}+\ldots+\frac{gv}{\lambda-\chi}\right),$$

so that $f(\lambda)$ is a cubic polynomial in λ in which the coefficient of λ^3 is $\left(\frac{axx_2}{x_1}\right)$; then we can shew that

$$\frac{a^2[f_1'(\xi)]^2}{f_2(\xi)} = \frac{b^2[f_1'(\eta)]^2}{f_2(\eta)} = \frac{c^2[f_1'(\zeta)]^2}{f_2(\zeta)} = \frac{f^2[f_1'(\omega)]^2}{f_2(\omega)} = \frac{g^2[f_1'(\chi)]^2}{f_2(\chi)}$$

$$= \left(\Sigma\frac{ax_2^2}{x_1}\right)\left(\Sigma\frac{ax_2}{x_1^2}\right),$$

for we find at once, from the equations satisfied by ρ, σ and by θ, ϕ, that

$$f_1(\lambda)\Sigma\frac{ax_2}{\lambda-\xi} = \lambda(\lambda-\rho)(\lambda-\sigma)\Sigma\frac{ax_2^2}{x_1}, \quad f_1(\lambda)\Sigma\frac{ax_1^{-1}}{\lambda-\xi} = \lambda(\lambda-\theta)(\lambda-\phi)\Sigma\frac{ax_2}{x_1^2},$$

and if we substitute $\lambda = \xi, \ldots,$ and multiply these equations, the results follow.

By the theory of partial fractions we have

$$\frac{f^3(\lambda)}{f_1(\lambda)f_2(\lambda)} = \Sigma_\xi\frac{f^3(\xi)}{(\lambda-\xi)f_1'(\xi)f_2(\xi)} + \Sigma_\alpha\frac{f^3(\alpha)}{f_1(\alpha)f_2'(\alpha)(\lambda-\alpha)},$$

where in the summation in regard to ξ, we are to put in turn ξ, η, ζ, ω, χ for ξ, and in the summation in regard to α, we are to put in turn 0, θ, ϕ, ρ, σ for α. Multiplying this equation by λ, and making λ infinite, we hence obtain

$$\left(\Sigma\frac{axx_2}{x_1}\right)^3 = \Sigma_\xi a\frac{a^2[f_1'(\xi)]^2}{f_2(\xi)}x^3 - \frac{f_1^2(\rho)}{f_2'(\rho)}\left(\Sigma\frac{axx_5}{x_2}\right)^3 - \frac{f_1^2(\sigma)}{f_2'(\sigma)}\left(\Sigma\frac{axx_6}{x_2}\right)^3$$

$$-\frac{f_1^2(0)}{f_2'(0)}\left(\Sigma\frac{axx_1}{x_2}\right)^3 - \frac{f_1^2(\theta)}{f_2'(\theta)}\left(\Sigma\frac{axx_1}{x_3}\right)^3 - \frac{f_1^2(\phi)}{f_2'(\phi)}\left(\Sigma\frac{axx_1}{x_4}\right)^3;$$

this equation is of the form

$$(\Sigma ax^3)\left(\Sigma\frac{ax_2^2}{x_1}\right)\left(\Sigma\frac{ax_2}{x_1^2}\right)$$

$$=\left(\Sigma\frac{axx_2}{x_1}\right)^3+A_0\left(\Sigma\frac{axx_1}{x_2}\right)^3+A_\theta\left(\Sigma\frac{axx_1}{x_3}\right)^3+A_\phi\left(\Sigma\frac{axx_1}{x_4}\right)^3$$

$$+A_\rho\left(\Sigma\frac{axx_5}{x_2}\right)^3+A_\sigma\left(\Sigma\frac{axx_6}{x_2}\right)^3,$$

whereby it is shewn that the equation of the cubic surface is obtainable by equating to zero a sum of the cubes of the six planes which have been described.

Conversely, assuming this form of identity, and acting upon it with the operator $\Sigma\,\frac{1}{x_1}\frac{\partial}{\partial x}$, we obtain

$$\left(\Sigma\frac{ax^2}{x_1}\right)\left(\Sigma\frac{ax_2}{x_1}\right)\left(\Sigma\frac{ax_2}{x_1^2}\right)$$

$$=\left(\Sigma\frac{axx_2}{x_1}\right)^2\left(\Sigma\frac{ax_2}{x_1^2}\right)+A_\rho\left(\Sigma\frac{axx_5}{x_2}\right)^2\left(\Sigma\frac{ax_5}{x_1x_2}\right)+A_\sigma\left(\Sigma\frac{axx_6}{x_2}\right)^2\left(\Sigma\frac{ax_6}{x_1x_2}\right),$$

and this shews that the three planes

$$\Sigma\frac{axx_2}{x_1}=0,\quad \Sigma\frac{axx_5}{x_2}=0,\quad \Sigma\frac{axx_6}{x_2}=0$$

are a set of self-conjugate planes in regard to the cone

$$\Sigma\frac{ax^2}{x_1}=0,$$

which is the polar quadric, in regard to the cubic surface, of the point $(1/x_1)$; by putting in this last identity (x_5) for (x) we have an equation determining A_ρ in terms of the original points. Similarly for the other cones and the other coefficients. We have, therefore, the theorem:

If an arbitrary straight line be taken to cut the Hessian of the cubic surface in the four points 1, 2, 3, 4, and the corresponding points of the Hessian to these be 1′, 2′, 3′, 4′ respectively, then the quadric cones with vertices at 1, 2, 3, 4 which are the polar quadrics respectively of 1′, 2′, 3′, 4′ in regard to the cubic surface, have common a pair of conjugate planes intersecting in the line 1234; these two planes, and the faces of the

tetrahedron $1'2'3'4'$ form six planes $X_1, \ldots, X_6$, such that the cubic surface can be written

$$\sum_{r=1}^{6} A_r X_r^3 = 0.$$

I have also carried out the reduction to this form by the symbolical processes of modern algebra, but it does not appear necessary to repeat the proof.

It is to be noticed that the plane

$$\frac{axx_1}{x_2 - \lambda x_1} + \ldots + \frac{gvv_1}{v_2 - \lambda v_1} = 0,$$

which can be written as a cubic polynomial in λ, is that of a variable osculating plane of a cubic curve in space, and includes the five coordinate planes $x=0$, $y=0$, $z=0$, $u=0$, $v=0$, namely, by $\lambda = \frac{x_2}{x_1}$, ..., and also includes the six planes found above, namely, $\Sigma \frac{axx_2}{x_1} = 0$ by $\lambda = \infty$, $\Sigma \frac{axx_1}{x_2} = 0$ by $\lambda = 0$, $\Sigma \frac{axx_1}{x_3} = 0$ by $\lambda = \theta$, $\Sigma \frac{axx_1}{x_4} = 0$ by $\lambda = \phi$, $\Sigma \frac{axx_5}{x_2} = 0$ by $\lambda = \rho$, and $\Sigma \frac{axx_6}{x_2} = 0$ by $\lambda = \sigma$. The five Sylvester planes are thus common osculating planes of an infinity of cubic curves; it is as defined in this way that Reye deduces Sylvester's expression.

The Beltrami method here adopted can be employed also to prove the following theorem:—Considering the rational curve of order n in space of n dimensions, of which the simplest cases are the conic and the cubic curve in space, there exists a homogeneous linear relation with constant coefficients connecting the r-th powers of any $rn+2$-osculating $(n-1)$-folds of the curve; here r is an arbitrary positive integer, and the $(rn+2)$-osculating $(n-1)$-folds may be taken arbitrarily, the ratios of their coefficients being then determinate. But we may also take m of the $(n-1)$-folds arbitrarily, subject to $\frac{1}{2}(rn+3) \leqslant m \leqslant rn+2$, and at the same time assign the ratios of $rn+3-m$ $(\leqslant m)$ of the coefficients associated therewith; then the remaining $(n-1)$-folds and their coefficients are determinable. For instance, for $n=3$, $r=3$ there is a relation connecting the cubes of eleven osculating planes of a cubic curve in space of three dimensions in which we may assign arbitrarily m of the osculating planes among those of the given cubic, provided $6 \leqslant m \leqslant 11$, with the ratios of $12-m$ of the coefficients of these. See also a paper, "On Canonical Forms," *Quarterly Journal*, Vol. XXXIV (1902), p. 117, by Mr. Richmond, by whom my attention was first called to Beltrami's method.

13. *On the Relation of the Lines of the Cubic Surface to the Hessian.*

If P, Q be two points in which a line lying on the cubic surface cuts the Hessian surface, it is easy to see that the tangent plane of the Hessian at either of these points passes through the other, so that PQ touches the Hessian at P and Q, and further, that these are corresponding points of the Hessian. Writing the cubic surface in the form $\Sigma ax^3 = 0$, with $\Sigma ax = 0$, which, as before, may be written $(ax^3) = 0$, with $(ax) = 0$, so that the Hessian is $(ax^{-1}) = 0$, the conditions that the line joining (x_1), (x_2) should lie on the cubic surface, namely,

$$(ax_1^3) = 0, \qquad (ax_1^2 x_2) = 0, \qquad (ax_1 x_2^2) = 0, \qquad (ax_2^3) = 0,$$

shew that the polar quadric of (x_2) in regard to the cubic surface, as well as the tangent plane of the cubic surface at (x_2), pass through (x_1); as the polar quadric is necessarily a cone, (x_2) being on the Hessian, it follows that this cone is touched by this plane all along the line joining (x_1) and (x_2), and has its vertex on this line; as the vertex must be on the Hessian, it is at (x_1). Thus, in virtue of $x_1x_2 = y_1y_2 = \ldots$, the above equations lead to

$$(ax_1^{-3}) = 0, \qquad (ax_2^{-3}) = 0,$$

as well as to the fact that the tangent plane of the Hessian at either of the points (x_1), (x_2) passes through the other. The equation

$$(ax^{-3}) = 0$$

is of the twelfth order, and the surface represented by it cuts the Hessian in a locus of order 48; this contains each of the ten lines lying upon the Hessian three times, leaving a locus of order 18, which cuts the cubic surface in $3.18 = 2.27$ points, namely, two on each of the twenty-seven lines of the cubic surface, as we have seen.

When the coefficients a, b, c, f, g are real, any real line of the cubic surface cuts the Hessian in points for which the values of any one coordinate are of the form $\xi+i\eta$ (twice) and $\xi-i\eta$ (twice); as the two distinct points are such that $x_1x_2 = y_1y_2 = \ldots$, it may be arranged that the coordinates shall be all of modulus unity, say

$$x : y : z : u : v = e^{i\theta_1} : e^{i\theta_2} : e^{i\theta_3} : e^{i\theta_4} : 1 ;$$

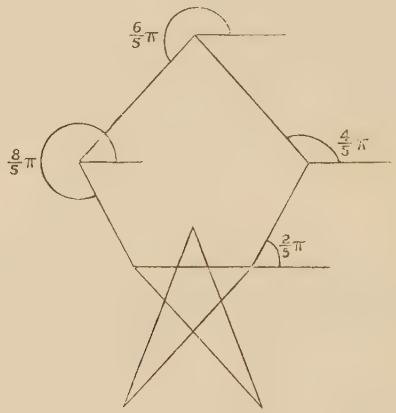
then the equations for these points,

$$(ax) = 0, \qquad (ax^{-1}) = 0, \qquad (ax^3) = 0, \qquad (ax^{-3}) = 0,$$

shew that five vectors $ae^{i\theta_1}$, $be^{i\theta_2}$, $ce^{i\theta_3}$, $fe^{i\theta_4}$, g, drawn from the same point,

have a zero sum, and that if the phases of these vectors be each multiplied by 3, their magnitudes being left unchanged, the new vectors still have sum zero. Or, in other words, that a closed polygon can be constructed, of sides a, b, c, f, g, with the property that, if the inclinations to one of the sides of all the other sides be tripled, a new closed polygon can be constructed with sides of the same length. And, conversely, to every closed polygon that can be constructed having this property with sides of length a, b, c, f, g, there corresponds a straight line of the cubic surface $(ax^3) = 0$, where $(ax) = 0$. The determination of the lines of the surface is thus reduced to a problem in elementary geometry.

For example, for the surface $x^3 = 0$, subject to $x = 0$, two such polygons are the pentagons indicated in the figure.



More generally, for the surface

$$x^3+y^3+z^3+u^3+mv^3 = 0,$$

with

$$x+y+z+u+mv = 0,$$

putting

$$(y+z)(z+x)(x+y) = \tfrac{1}{3}[(x+y+z)^3-(x^3+y^3+z^3)] = -mv[u+\alpha v][u+\beta v],$$

where

$$\alpha+\beta = m, \qquad \alpha\beta = \tfrac{1}{3}(m^2-1),$$

and proceeding, by the methods previously explained in this paper, to find the lines of the cubic surface, and thence their points of contact with the Hessian, we find one such point of contact to be given by

$$x : y : z : u : v = e^{i\lambda} : e^{-i\lambda} : e^{i\alpha} : e^{-i\alpha} : 1,$$

where

$$\cos\alpha = \tfrac{1}{4}\left[\left(\frac{16-m^2}{3}\right)^{\frac{1}{2}} - m\right], \quad \cos\lambda = -\tfrac{1}{4}\left[\left(\frac{16-m^2}{3}\right)^{\frac{1}{2}} + m\right],$$

this corresponding to five vectors $e^{i\lambda}$, $e^{-i\lambda}$, $e^{i\alpha}$, $e^{-i\alpha}$, m. Fifteen of the lines of the cubic surface meet pairs of lines such as $y = 0$, $z = 0$ and $u = 0$, $v = 0$; the remaining twelve lines are determined by the twenty-four possible permutations of the values here given for x/v, y/v, z/v, u/v.

APPENDIX.

On the Construction of a Double Six.

If an arbitrary line of space, a_1, be taken, and five non-intersecting lines intersecting this, say b_2, b_3, b_4, b_5, b_6; if, then, a_2 be the transversal other than a_1 which can be drawn to intersect b_3, b_4, b_5, b_6, and a_3 be the transversal other than a_1 which can be drawn to meet b_2, b_4 b_5, b_6,, and so on, then the five lines a_2, a_3, a_4, a_5, a_6 so found have a common transversal, b_1, which does not intersect a_1.

This is a proposition of elementary geometry, obviously true because a definite cubic can be drawn to contain a_1 and b_2, b_3, b_4, b_5, b_6, but capable of proof independent of such surface. A direct algebraic proof is given below, so formulated as to shew that the theorem belongs to three dimensions only (§ III). This is very interesting as a study in line-geometry, but of length disproportionate to the geometrical theorem, and is given only in abstract. But a geometrical proof of the existence of twelve lines related as above* was given by Schur (*Math. Annalen*, Bd. XVIII, 1881, pp. 10, 11), based on the assumption of a poristic property of a plane cubic curve, and suggested by a particular theorem to which he refers (Vogt, *Crelle*, Bd. LXXXVI, p. 301). My own knowledge of Schur's method began with a communication of Mr. G. T. Bennett to the London Mathematical Society in January, 1910; Mr. Bennett had rediscovered, analytically, the necessary invariant relation connecting two quadric surfaces which arise in the proof. There are also papers by Prof. Burnside (*Camb. Phil. Proceedings*, Lent Term, 1910) and by Prof. A. C. Dixon (*Quarterly Journal*, Vol. XL, 1909, pp. 381–384); Mr. Grace has also sketched a proof of Schur's theorem (see Jessop's "Treatise on the Line Complex," 1903, Example 40, p. 354), and Mr. Richmond made a communication to the London Mathematical Society in regard to the matter in February, 1910. All these I have had the advantage of, and what follows in § I is, in fact, a joint communication of these authors; but even so, and though the fundamental idea is undoubtedly due to Schur, it would seem that there is enough of novelty to justify its publication. In § II a consideration of Schur's method applied to an elliptic

* [*Dec.* 17*th*, 1910.—A simple geometrical proof, independent of the ideas of the text, will be found in a paper read before the Royal Society on November 24th, to be published shortly in the *Proceedings of the Royal Society*.]

quartic curve in space is given. I understand that, so far as relates to the existence of a double six, Mr. Richmond has communicated to the Society some remarks in this regard; I have ventured to retain what I had written before Mr. Richmond's communication in order to call attention to certain triads of double sixes, not all lying on the same cubic surface, which are very similar to some triads of double sixes known to lie on the same cubic surface.

This long apology for the §§ I and II following seems to be the only alternative to suppressing them altogether, which I am unwilling to do because the matter appears very interesting and may appeal to others as deserving further study.

I. *The Elementary View of Schur's Figure.*

Given a perfectly arbitrary skew hexagon, whose sides in order are named a, b', c, a', b, c', a line meeting the alternate sides a, b, c can be determined by drawing an arbitrary plane, say $P = \sigma Q$, through a; such a line, therefore, depends rationally on one parameter σ; in the same way a line meeting the other set of alternate sides a', b', c' may be supposed determined rationally by a parameter σ', by means of a plane through a'. Any arbitrary line σ_1 being taken to intersect a, b, c, two lines, say σ_2' and σ_3', can be drawn to intersect a', b', c', σ_1; then two lines, σ_1, σ_3 can be drawn to meet a, b, c, σ_2', and two lines, σ_1, σ_2, can be drawn to meet a, b, c, σ_3'. Now regard two lines which meet a, b, c as corresponding to one another when they are both met by a transversal of a', b', c'; then it is clear that to σ_1 correspond only the two lines σ_2, σ_3; namely, σ_2, σ_3 are the roots of a quadratic equation whose coefficients are rational in σ_1. Moreover, the correspondence of two lines meeting a, b, c, as defined, is symmetrical; if σ_1 correspond to σ_2 (and σ_3), then σ_1 is one of the lines corresponding to σ_2. The relation connecting two such corresponding lines is thus of the form

$$\sigma_1^2(a\sigma_2^2+h\sigma_2+g)+\sigma_1(h\sigma_2^2+b\sigma_2+f)+g\sigma_2^2+f\sigma_2+c = 0.$$

This may be interpreted as the condition that the points $(\sigma_1^2, \sigma_1, 1)$, $(\sigma_2^2, \sigma_2, 1)$ of the conic $xz-y^2 = 0$ should be conjugate in regard to the conic $(a, b, c, f, g, h\mathcal{X}x, y, z)^2 = 0$. Thus the condition that when σ_1 corresponds to σ_2 and σ_3, the two lines corresponding to σ_2 should be σ_1 and σ_3, is that it be possible to inscribe a triangle in the former conic $xz-y^2 = 0$, self-polar in regard to $(a, b, c, f, g, h\mathcal{X}x, y, z)^2 = 0$, namely, $B = G$ or $ca-g^2 = hf-bg$; and, when this is satisfied, the first line σ_1 may be taken arbitrarily, the other two lines σ_2, σ_3 following without ambiguity; with these are determined three other lines σ_1', σ_2', σ_3' meeting

a', b', c', so that σ_1, σ_2', σ_3, σ_1', σ_2, σ_3' form a new skew hexagon, of which either set of alternate sides all meet one set of alternate sides of the first hexagon. These twelve lines together will form a double six, which is thus determined by an arbitrary skew hexagon and any line meeting one set of alternate sides of this.

That the necessary condition is really satisfied is obvious, however; for, if the quadrics determined respectively by a, b, c and a', b', c' be regarded under the forms

$$x^2/p+y^2/q-z^2/r = 1, \qquad x^2/p'+y^2/q'-z^2/r' = 1,$$

the generators a, b' of these project orthogonally (from $x = 0 = y$, $z = \infty$) into intersecting generators A, B' of the opposite systems of these quadrics respectively; by this projection we thus have another skew hexagon A, B, C, A', B', C', of which A, B, C meet a, b, c, and A', B', C' meet a', b', c'. And a similar projection is possible from each of the angular points of the common self-polar tetrahedron of these quadrics.

It can easily be shewn that a definite quadric surface Σ exists in regard to which opposite sides of a skew hexagon a, b', c, a', b, c are polars of one another. It is not difficult to see that, if we have an interlaced hexagon σ_1, σ_2', σ_3, σ_1', σ_2, σ_3' such as that above, either set of alternate sides of this meeting one set of alternate sides of $ab'ca'bc$, then opposite sides of this are also polars in regard to Σ. This is Schur's theorem. In regard to Σ, the relation of the three lines a, b, c is that any two are conjugate, namely, each meets the polar of the other taken in regard to Σ. Denoting by S the quadric surface determined by the three non-intersecting lines a, b, c, Mr. Bennett has found that, in order for three generators of a quadric S to be in such relation in regard to a quadric Σ, the necessary invariant condition is $\Theta\Theta' = 4\Delta\Delta'$, where Δ, Θ, Θ', Δ' are the invariants of S, Σ.

It may be worth while to indicate how to proceed analytically directly on the lines of the general argument for the existence of a double six given above. If $AB'CA'BC'$ be the vertices of a skew hexagon, and the four planes $AB'C$, $B'CA'$, $A'BC'$, $BC'A$ be taken respectively for $z = 0$, $y = 0$, $x = 0$, $t = 0$, the angular points of the hexagon having coordinates

$$A\,(a, 1, 0, 0), \qquad B\,(0, 1, b, 0), \qquad C\,(1, 0, 0, c),$$
$$A'(0, 0, a', 1), \qquad B'(b', 0, 0, 1), \qquad C'(0, c', 1, 0),$$

the quadric in regard to which opposite sides of the hexagon are polars of one another is

$$a'(cx^2+b't^2-2xt) = a(by^2+c'z^2-2yz);$$

putting $\xi = cx-t$, $\eta = by-z$, $\zeta = z$, $\tau = t$,

this is $$A\xi^2+B\eta^2+C\zeta^2+D\tau^2=0,$$

where $A=a'$, $B=-ca/b$, $C=ca(1-bc')/b$, $D=-(1-b'c)a'$.

With these coordinates, if we put $\sigma=nD/A^2Ba'$, the general line, say (l, m, n, l', m', n'), meeting the sides BC', CA', AB' of the hexagon is found to be

$$l=0, \qquad m=-\frac{\sigma}{1-\sigma}\frac{A(1+AB)}{D}, \qquad n=\frac{\sigma A^2B}{D},$$

$$l'=\frac{A(1+AB\sigma)}{1+AB}, \qquad m'=\frac{AB(1-\sigma)}{1+AB}, \qquad n'=1.$$

Similarly, with $\rho=\nu'/AC\nu$, the general line $(\lambda, \mu, \nu, \lambda', \mu', \nu')$, meeting $B'C$, $C'A$, $A'B$, is

$$\lambda=\frac{C(1+AB\rho)}{1+AB}, \qquad \mu=\frac{AC(1-\rho)}{1+AB}, \qquad \nu=1,$$

$$\lambda'=0, \qquad \mu'=-\frac{\rho}{1-\rho}(1+AB), \qquad \nu'=\rho AC.$$

The condition for these lines $(l, \ldots)$, $(\lambda, \ldots)$ to intersect reduces to

$$(1-\rho)(1-\sigma)(1+u\rho\sigma)=v\rho\sigma,$$

where $$u=A^2BC(A+D+A^2B)/D(1+AB+AC),$$
$$v=A(1+AB)^3/D(1+AB+AC).$$

From this relation, if ρ_2, ρ_3 be the values of ρ corresponding to the values σ_1 of σ, and so on, we find

$$\rho_1=\sigma_1, \qquad \rho_2=\sigma_2, \qquad \rho_3=\sigma_3;$$

and these verify that the six pairs of lines, the original sides of the hexagon and the six $\rho_1, \ldots, \sigma_3$, form pairs of polars in regard to the quadric.

The fact that when a skew hexagon is constituted by six lines $ab'ca'bc'$ an infinite number of such hexagons exist of which one set of alternate sides is formed by generators of the quadric determined by a, b, c, and the other set of alternate sides by generators of the quadric determined by a', b', c', is also at once obvious thus. Let the line coordinates of a be $a_1, a_2, \ldots, a_6$, subject to $a_1^2+a_2^2+\ldots+a_6^2=0$, and so on; and let (ab') denote $a_1b_1'+a_2b_2'+\ldots+a_6b_6'$, and so on. The conditions assumed are then

$$(ab')=(ac')=(bc')=(ba')=(ca')=(cb')=0.$$

The line coordinates of the general line of the quadric (a, b, c) of the same system of generators as these are linear functions of the line coordinates of these; thus we may represent three generators of (a, b, c) and three

generators of (a', b', c') respectively by

$$(A) = \xi_1(a)+\eta_1(b)+\zeta_1(c), \qquad (A') = \xi_1'(a')+\eta_1'(b')+\zeta_1'(c'),$$
$$(B) = \xi_2(a)+\eta_2(b)+\zeta_2(c), \qquad (B') = \xi_2'(a')+\eta_2'(b')+\zeta_2'(c'),$$
$$(C) = \xi_3(a)+\eta_3(b)+\zeta_3(c), \qquad (C') = \xi_3'(a')+\eta_3'(b')+\zeta_3'(c'),$$

where each equation represents six equations, one for each of the coordinates; that these should represent lines, it is necessary that

$$\frac{f}{\xi_1}+\frac{g}{\eta_1}+\frac{h}{\zeta_1}=0, \quad \frac{f}{\xi_2}+\frac{g}{\eta_2}+\frac{h}{\zeta_2}=0, \quad \frac{f}{\xi_3}+\frac{g}{\eta_3}+\frac{h}{\zeta_3}=0,$$

and $$\frac{f'}{\xi_1'}+\frac{g'}{\eta_1'}+\frac{h'}{\zeta_1'}=0, \quad \frac{f'}{\xi_2'}+\frac{g'}{\eta_2'}+\frac{h'}{\zeta_2'}=0, \quad \frac{f'}{\xi_3'}+\frac{g'}{\eta_3'}+\frac{h'}{\zeta_3'}=0,$$

where $$f = (bc), \qquad g = (ca), \qquad h = (ab),$$
$$f' = (b'c'), \qquad g' = (c'a'), \qquad h' = (a'b');$$

that (A) should intersect (B') and (C'), and so on, or, in other words, in order that $AB'CA'BC'$ should also form a skew hexagon, there must hold six conditions such as the two

$$u\xi_1\xi_2'+v\eta_1\eta_2'+w\zeta_1\zeta_2' = 0,$$
$$u\xi_1\xi_3'+v\eta_1\eta_3'+w\zeta_1\zeta_3' = 0,$$

where $$u = (aa'), \qquad v = (bb'), \qquad w = (cc').$$

Geometrically, interpreting (ξ_1, η_1, ζ_1), ..., as coordinates of points in a plane, it must be possible to inscribe a triangle (1, 2, 3) in the conic $f/\xi+g/\eta+h/\zeta = 0$ whose polar triangle (1′, 2′, 3′), in regard to

$$u\xi^2+v\eta^2+w\zeta^2 = 0,$$

should be inscribed in $f'/\xi+g'/\eta+h'/\zeta = 0$. This is evidently a porismatic property for which the necessary conditions are satisfied, one such figure being obtainable by

$$\eta_1 = \eta_1' = \zeta_1 = \zeta_1' = 0, \qquad \zeta_2 = \zeta_2' = \xi_2 = \xi_2' = 0, \qquad \xi_3 = \xi_3' = \eta_3 = \eta_3' = 0.$$

II. *Double Sixes and a Skew Quartic Curve of the First Species.*

If in two quadric surfaces

$$P \equiv y^2+e_1X^2+e_2Z^2+e_3T^2 = 0, \qquad Q \equiv X^2+Z^2+T^2 = 0,$$

where e_1, e_2, e_3 are the roots of $4\theta^3-g_2\theta-g_3 = 0$, we put

$$\xi = \tfrac{1}{2}X[(e_1-e_2)(e_1-e_3)]^{-\frac{1}{2}}, \qquad \zeta = \tfrac{1}{2}Z[(e_2-e_3)(e_2-e_1)]^{-\frac{1}{2}},$$
$$\tau = \tfrac{1}{2}T[(e_3-e_1)(e_3-e_2)]^{-\frac{1}{2}},$$

and then

$$x = e_1\xi + e_2\zeta + e_3\tau, \qquad z = (g_2 - 12e_1^2)\xi + (g_2 - 12e_2^2)\zeta + (g_2 - 12e_3^2)\tau,$$

$$t = \xi + \zeta + \tau,$$

we find $P = y^2 - \frac{2}{3}xz + \frac{2}{3}g_2xt + g_3t^2$, $Q = 4x^2 - \frac{2}{3}zt - \frac{1}{3}g_2t^2$;

thus the quartic curve $P = 0$, $Q = 0$ is satisfied by

$$\frac{x}{t} = \wp(u), \qquad \frac{y}{t} = \wp'(u), \qquad \frac{z}{t} = \wp''(u),$$

and, in fact, the substitutions

$$\frac{x}{t} = \lambda\wp(u) + (1-\lambda)\wp(v), \qquad \frac{y}{t} = \lambda\wp'(u) + (1-\lambda)\wp'(v),$$

$$\frac{z}{t} = \lambda\wp''(u) + (1-\lambda)\wp''(v)$$

lead, independently of the value of λ, to

$$\wp(u+v) = P/Q,$$

and therefore any chord of the quartic curve joining points u, v of the curve for which $u+v$ is constant, equal to a, say, lies upon the quadric surface $P - Q\wp(a) = 0$, containing the quartic curve; the same quadric also contains the chords joining points of arguments of sum $-a$, these being the other system of generators, and the quadric may be called the quadric (a).

The points of the quartic curve lying on an arbitrary plane are such that the sum of their four elliptic arguments is zero, save for multiples of the period.

Now consider any two arbitrary chords of the quartic curve, the sum of the arguments for the extremities of the first chord being $-p$, and for the second $-p'$. An arbitrary point of the quartic being joined by a plane to the first chord, this plane will cut the quartic in one further point, namely, a definite chord of the curve can be drawn from an arbitrary point (A) of the curve to meet the first chord; this will cut the curve again in (B'), where $A + B' \equiv p$. From (B') a chord passes to cut the second given chord $(-p')$, cutting the curve again in (C), where $B' + C \equiv p'$. Proceeding thus alternately with the two chords, we have equations

$$A + B' \equiv p, \qquad B' + C \equiv p', \qquad C + A' \equiv p,$$

$$A' + B \equiv p', \qquad B + C' \equiv p, \qquad C' + A_1 \equiv p',$$

and, if the process terminates, forming a skew hexagon whose sides alternately intersect the chords p, p', namely, if $A_1 \equiv A$, we have

$3p' \equiv 3p$. And conversely this condition is sufficient to ensure the termination.

Any chord intersecting $(-p)$ will also intersect all chords whose end points have arguments of sum $-p$, that is, all those belonging to the quadric generated by chords $(-p)$. Thus we may suppose the two original chords to be taken to have a common arbitrary intersection on the quartic curve; the other extremities of these chords may then be defined as any two points whereat the osculating planes of the curve co-intersect again on the curve.

If we put $p' = p + \frac{1}{3}\Omega$, where Ω is a period of the elliptic functions, we find

$$A \equiv A, \qquad A' \equiv p - A + \tfrac{2}{3}\Omega,$$
$$B \equiv A + \tfrac{2}{3}\Omega, \qquad B' \equiv p - A,$$
$$C \equiv A + \tfrac{1}{3}\Omega, \qquad C' \equiv p - A + \tfrac{1}{3}\Omega.$$

The sides BC', CA', AB' belong to the quadric (p), the sides $B'C$, $C'A$, $A'B$ belong to the quadric $(p + \frac{1}{3}\Omega)$. Taking an arbitrary point of the quartic, we can form another skew hexagon of which three alternate sides belong to one quadric $(-p)$ and intersect BC', CA', AB', the other set of alternate sides intersecting $B'C$, $C'A$, $A'B$ and belonging to the quadric $(-p + \frac{2}{3}\Omega)$. These two quadrics are the same as the two former. The twelve lines so described make a double six.

If we begin the new hexagon from a point (α) of the quartic curve, and name $\alpha\beta'$ a chord drawn to intersect AB', BC', CA', the successive corners of this second hexagon are determined by the equations

$$\alpha + \beta' \equiv -p, \qquad \beta' + \gamma \equiv -p - \tfrac{1}{3}\Omega, \qquad \gamma + \alpha' \equiv -p,$$
$$\alpha' + \beta \equiv -p - \tfrac{1}{3}\Omega, \qquad \beta + \gamma' \equiv -p, \qquad \gamma' + \alpha \equiv -p - \tfrac{1}{3}\Omega,$$

leading to

$$\alpha \equiv \alpha, \qquad \alpha' \equiv -p - \alpha + \tfrac{1}{3}\Omega,$$
$$\beta \equiv \alpha + \tfrac{1}{3}\Omega, \qquad \beta' \equiv -p - \alpha,$$
$$\gamma \equiv \alpha + \tfrac{2}{3}\Omega, \qquad \gamma' \equiv -p - \alpha + \tfrac{2}{3}\Omega.$$

From these forms various results are deducible. We have

$$A + A' \equiv B + B' \equiv C + C' \equiv -(\alpha + \alpha') \equiv -(\beta + \beta') \equiv -(\gamma + \gamma') \equiv p + \tfrac{2}{3}\Omega;$$

thus the joins of opposite angular points of the original hexagon define a quadric surface containing the quartic curve and independent of the position of (A); this contains, as three generators of the opposite system, the joins of opposite angular points of the second hexagon defined by the point (α). We thus have the theorem of elementary geometry: The quadric surface defined by the joins of opposite angular points of an arbitrary skew hexagon, passes through the curve of intersection of the

quadrics defined by the two sets of three alternate sides of the hexagon; and, if another hexagon be taken, interlaced with the given one, so that either set of three alternate sides of one hexagon meets three alternate sides of the other, the joins of opposite angular points of the new hexagon are also generators of the quadric first mentioned.

As we have seen, the quadrics defined by the alternate sides and by the diagonals of the original hexagon are (p), $(p+\frac{1}{3}\Omega)$, $(p+\frac{2}{3}\Omega)$. The points $AA'BB'CC'$ form a new skew hexagon whose alternate sides generate the quadrics $(p+\frac{2}{3}\Omega)$, $(p+\frac{1}{3}\Omega)$; these are alternately intersected by the alternate sides of the skew hexagon $a\alpha'\beta\beta'\gamma\gamma'$. Thus beside the double six formed by the two hexagons $AB'CA'BC'$, $a\beta'\gamma a'\beta\gamma'$, we have the double six formed by $AA'BB'CC'$ and $a\alpha'\beta\beta'\gamma\gamma'$. A third double six is given by the interlaced hexagons $AA'CC'BB'$ and $a\alpha'\gamma\gamma'\beta\beta'$, whose sides lie alternately on the quadrics $(p+\frac{2}{3}\Omega)$, (p). It is at once seen that a cubic surface can be described to contain the sides of a skew hexagon together with an arbitrary point; in fact, if for such a hexagon $AB'CA'BC'$ the planes $AB'C$, $B'CA'$, $CA'B$, $A'BC'$, $BC'A$, $C'AB'$ be $\eta'=0$, $\xi=0$, $\xi'=0$, $\eta=0$, $\zeta'=0$, $\xi=0$, the cubic surface containing the hexagon is $\xi\eta\zeta+\theta\xi'\eta'\xi'=0$. We can thus describe three cubic surfaces containing the three double sixes which have been introduced, and it is at once seen that any two of these cubic surfaces have six lines common lying upon an hyperboloid, so that the remaining intersection of such two surfaces is a locus of the third order.

Let us now denote the various lines as follows:

BC' by a_1, CA' by a_2, AB' by a_3, $\beta'\gamma$ by a_4, $\gamma'a$ by a_5, $a'\beta$ by a_6,

$B'C$ by b_1, $C'A$ by b_2, $A'B$ by b_3, $\beta\gamma'$ by b_4, $\gamma a'$ by b_5, $a\beta'$ by b_6,

AA' by c'_{23}, BB' by c'_{31}, CC' by c'_{12}, $a\alpha'$ by c'_{56}, $\beta\beta'$ by c'_{64}, $\gamma\gamma'$ by c'_{45},

$$\left.\begin{matrix} AB'C' \\ A'BC \end{matrix}\right\} \text{ by } c_{23}, \quad \left.\begin{matrix} BC'A' \\ B'CA \end{matrix}\right\} \text{ by } c_{31}, \quad \left.\begin{matrix} CA'B' \\ C'AB \end{matrix}\right\} \text{ by } c_{12},$$

$$\left.\begin{matrix} a\beta'\gamma' \\ a'\beta\gamma \end{matrix}\right\} \text{ by } c_{56}, \quad \left.\begin{matrix} \beta\gamma'a' \\ \beta'\gamma a \end{matrix}\right\} \text{ by } c_{64}, \quad \left.\begin{matrix} \gamma a'\beta' \\ \gamma' a\beta \end{matrix}\right\} \text{ by } c_{45},$$

where, for instance, $\left.\begin{matrix} AB'C' \\ A'BC \end{matrix}\right\}$ means the line of intersection of the planes $AB'C'$ and $A'BC$, so that c_{23} is the line of intersection of the plane of the intersecting lines a_3, b_2 with the plane of the intersecting lines a_2, b_3. Then what we have shewn is (a) that

$$\left.\begin{matrix} a_1, & a_2, & a_3, & a_4, & a_5, & a_6 \\ b_1, & b_2, & b_3, & b_4, & b_5, & b_6 \end{matrix}\right\} \qquad \text{(I)}$$

are a double six, determined essentially by the arbitrary skew hexagon $AB'CA'BC'$ and the arbitrary point (α) upon the quartic curve of intersection of the two quadrics defined by the two sets of alternate sides of this hexagon. Denoting by $(AB'C) = 0$ the equation of the plane through three points A, B', C, this double six lies on the cubic surface

$$(A'BC)(B'CA)(C'AB) = \lambda(AB'C')(BC'A')(CA'B'),$$

λ being determined so that this passes through the point (α). This also shews incidentally that when, as here, we have two interlaced hexagons $AB'CA'BC'$, $\alpha\beta'\gamma\alpha'\beta\gamma'$, there exists an identity of the form

$$(A'BC)(B'CA)(C'AB)+\lambda(AB'C')(BC'A')(CA'B')+\mu(\alpha'\beta\gamma)(\beta'\gamma\alpha)(\gamma'\alpha\beta)$$
$$+\rho(\alpha\beta'\gamma')(\beta\gamma'\alpha')(\gamma\alpha'\beta') = 0.$$

(*b*) That the following twelve lines also constitute a double six

$$\left.\begin{matrix} a_1, & a_2, & a_3, & c'_{56}, & c'_{64}, & c'_{45} \\ c'_{23}, & c'_{31}, & c'_{12}, & b_4, & b_5, & b_6 \end{matrix}\right\}, \qquad \text{(II)}$$

similarly associated with the hexagon $AA'CC'BB'$, and lying on a cubic surface of the form

$$(AA'B')(BB'C')(CC'A') = p(AA'C)(BB'A)(CC'B).$$

(*c*) That the following twelve lines are also a double six

$$\left.\begin{matrix} c'_{23}, & c'_{31}, & c'_{12}, & a_4, & a_5, & a_6 \\ b_1, & b_2, & b_3, & c'_{56}, & c'_{64}, & c'_{45} \end{matrix}\right\}, \qquad \text{(III)}$$

similarly associated with the hexagon $AA'BB'CC'$, and lying on a cubic surface of the form

$$(AA'B)(BB'C)(CC'A) = q(AA'C')(BB'A')(CC'B').$$

There are utilised in these three double sixes only eighteen lines, every two of the double sixes having six lines in common, which are generators of a hyperboloid. These three hyperboloids have a common quartic curve; the three cubic surfaces are distinct. It is interesting to compare the figure with the fact that upon the same cubic surface there exist threes of double sixes of which every two have six lines in common. For instance, upon the first of the cubic surfaces here referred to we have the three double sixes

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{matrix} \Big| \begin{matrix} a_1 & a_2 & a_3 & c_{56} & c_{64} & c_{45} \\ c_{23} & c_{31} & c_{12} & b_4 & b_5 & b_6 \end{matrix} \Big| \begin{matrix} c_{23} & c_{31} & c_{12} & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & c_{56} & c_{64} & c_{45} \end{matrix};$$

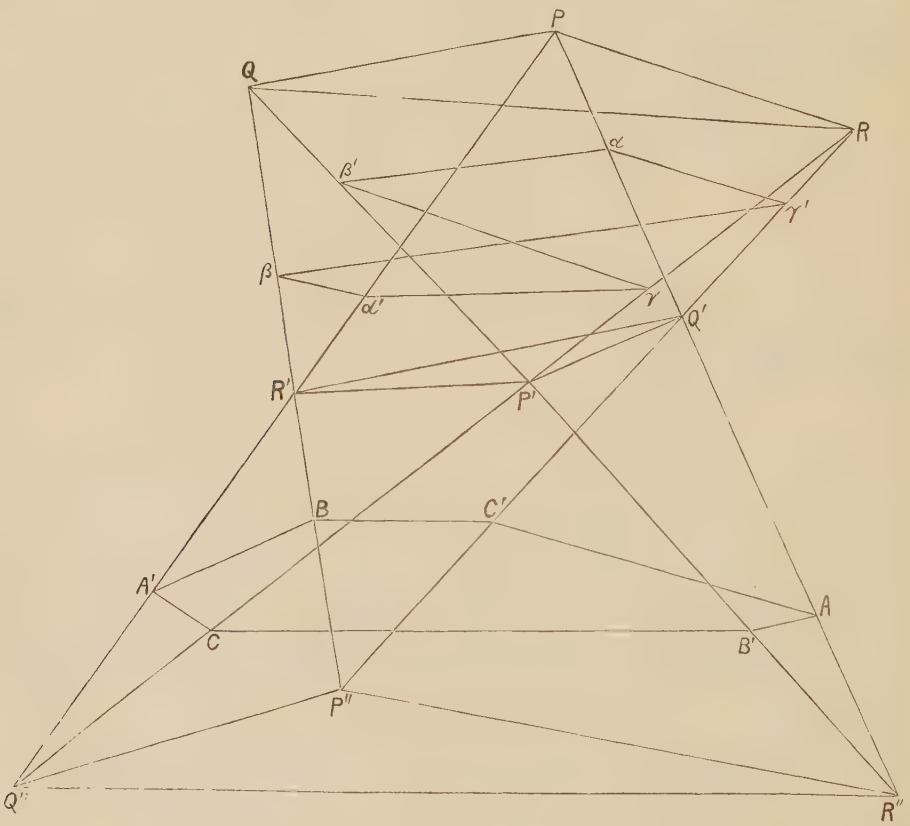
similarly for each of the other two cubic surfaces the double sixes put down enable us to name two others for each of these surfaces.

It will be noticed that whereas c_{ij} is the intersection of the planes of a_i, b_j and a_j, b_i, the line c'_{ij} has the dual definition, being the join of the points of intersection a_i, b_j and a_j, b_i; we have remarked that c'_{23}, c'_{31}, c'_{12} generate a quadric passing through the intersection of the quadrics (a_1, a_2, a_3), (b_1, b_2, b_3). We infer therefore that: On any cubic surface, with the usual notation for the lines, the quadric surface defined by (c_{23}, c_{31}, c_{12}) touches the developable formed by the common tangent planes of the quadrics (a_1, a_2, a_3), (b_1, b_2, b_3). Or by a projection we can regard the quadrics (a_1, a_2, a_3), (b_1, b_2, b_3) and (c_{23}, c_{31}, c_{12}) as confocals.

Another relation following from the equations connecting the arguments of the points considered on the quartic curve is

$$A+\alpha \equiv B+\beta \equiv C+\gamma \equiv -(A'+\alpha') \equiv -(B'+\beta') \equiv -(C'+\gamma');$$

this shews that $A\alpha$, $B\beta$, $C\gamma$ and $A'\alpha'$, $B'\beta'$, $C'\gamma'$ are six generators of the same quadric, three of each system. We can thus draw the annexed figure, wherein $\alpha\beta'$ intersects BC' and CA', $\alpha\alpha'$ intersects BB' and CC',



$\alpha\beta'$ and BC' intersect on QQ', $\alpha\beta'$ and CA' intersect on PP', $\alpha\alpha'$ and BB' meet on $R'R''$, $\alpha\alpha'$ and CC' meet on $Q'Q''$, and so on.

Finally, it may be remarked, as a consequence of the equations

$$B \equiv A + \tfrac{2}{3}\Omega, \qquad C \equiv A + \tfrac{1}{3}\Omega,$$

that the osculating planes of the quartic curve at the points A, B, C meet in a point $(-3A)$ of the curve lying on the plane ABC. The osculating planes at A', B', C' similarly intersect at the point $-3p+3A$. The chord joining these two points lies on a quadric surface $(-3p)$, independent of the position of (A) on the quartic curve. There are twelve osculating planes passing through $(-3A)$, the points of osculation lying by threes on twelve planes such as ABC, all passing through $(-3A)$; and a similar remark holds in regard to osculating planes through $(-3p+3A)$. It would seem worth while to examine the complete figure obtained by taking any three of the first points of osculation with three of the second to form a hexagon.

III. *The direct Problem of the Double Six, as an Exercise in Analytical Line Geometry.*

Suppose we are given one line and five other non-intersecting lines intersecting this one; we know that six other lines can be put with these so as to form a double six; it is a problem of algebra to express the coordinates of these others in terms of those of the given lines, and to find the algebraic identity upon which the existence of a double six depends, and the work is interesting as a study of certain determinants.

By putting the ordinary Plücker line coordinates respectively equal to $(x+iu, y+iv, z+iw, x-iu, y-iv, z-iw)$, we may interpret the line as a point lying upon the quadric four-fold $x^2+y^2+z^2+u^2+v^2+w^2 = 0$ in five-dimensional space. We shall use this phraseology.

(1) Upon this quadric we take five points such as $(A_i, B_i, C_i, F_i, G_i, H_i)$; this point we denote by (A_i). The condition that the five lines of three-dimensional space which these points represent should meet another line, is that a point upon the linear four-fold determined by them, namely, a point with coordinates of the form

$$\theta_1 A_1 + \ldots + \theta_5 A_5, \qquad \theta_1 B_1 + \ldots + \theta_5 B_5, \qquad \ldots,$$

should be upon the quadric $(x^2) = 0$, and also upon the tangent four-folds

of this quadric at each of the points (A_i); denoting by c_{ij} the expression

$$c_{ij} = A_iA_j + B_iB_j + \ldots + H_iH_j,$$

the necessary and sufficient condition for this is that the determinant of five rows and columns

$$C_{00} = \begin{vmatrix} c_{11} & c_{12} & \ldots & c_{15} \\ \ldots & \ldots & \ldots & \ldots \\ c_{51} & c_{52} & \ldots & c_{55} \end{vmatrix}$$

should vanish; and then, denoting by ω_i^2, $\omega_i\omega_j$ the minors of the elements c_{ii}, c_{ij} in C_{00}, the five points (A_i) lie upon the tangent four-fold of the quadric at the point whose coordinates are

$$\omega_1A_1 + \ldots + \omega_5A_5, \qquad \omega_1B_1 + \ldots + \omega_5B_5, \qquad \ldots;$$

this point we denote by

$$(\xi) = (\omega_1A_1 + \ldots + \omega_5A_5).$$

In what follows all is expressed in terms of the elements of the determinant C_{00}, that is, in terms of five points connected by the relation $C_{00} = 0$; the points (A_i) and (ξ) are the six fundamental points referred to above.

(2) We now seek to draw the five tangent four-folds to the fundamental quadric passing respectively through four of the points (A_i), other than that at (ξ); and to express the condition that their point of intersection (ξ') lies upon the quadric. When this is secured the double six of points is determined, consisting of a "hexahedron" of four-folds which is both inscribed to and circumscribed about the fundamental quadric. From this point of view it may appear probable that a similar property holds for a quadric in any number of dimensions; in order to examine this the work is arranged so that only at the final step is the number of dimensions of any importance.

The general four-fold containing (A_1), (A_2), (A_3), (A_4) is of the form $P + \lambda Q$, and there are two values of λ for which it touches the quadric. Expressing that a point

$$(x + \theta_1A_1 + \theta_2A_2 + \theta_3A_3 + \theta_4A_4)$$

lies upon the quadric and is conjugate to each of (A_1), (A_2), (A_3), (A_4), so that the tangent four-fold at this point passes through each of these, we find as the equation of the two tangent four-folds of the quadric which

pass through (A_1), (A_2), (A_3), (A_4),

$$\begin{vmatrix} (x^2), & (xA_1), & (xA_2), & (xA_3), & (xA_4) \\ (xA_1), & c_{11}, & c_{12}, & c_{13}, & c_{14} \\ (xA_2), & c_{21}, & c_{22}, & c_{23}, & c_{24} \\ (xA_3), & c_{31}, & c_{32}, & c_{33}, & c_{34} \\ (xA_4), & c_{41}, & c_{42}, & c_{43}, & c_{44} \end{vmatrix} = 0,$$

where $c_{ii} = 0$, $c_{ij} = c_{ji}$, $(x^2) = x^2 + \ldots + w^2$, $(xA_1) = xA_1 + \ldots + wH_1$, ...

This determinant is the minor X_{55} of the determinant X, of six rows and columns,

$$X = \begin{vmatrix} (x^2), & (xA_1), & \ldots, & (xA_5) \\ (xA_1), & c_{11}, & \ldots, & c_{15} \\ \ldots & \ldots & \ldots & \ldots \\ (xA_5), & c_{51}, & \ldots, & c_{55} \end{vmatrix}.$$

The numerical value of X, since in it the minor of (x^2) is the vanishing determinant C_{00}, is

$$X = -[\omega_1(xA_1) + \ldots + \omega_5(xA_5)]^2 = -(x\xi)^2.$$

By what we have seen each of the principal minors $X_{11}, \ldots, X_{55}$ of X divides by $(x\xi)$; we have, in fact, found X_{55} as the equation of two four-folds of which $(x\xi)$ is one. The minor of (xA_1) in X is

$$X_{01} = -(xA_1)\omega_1^2 - (xA_2)\,\omega_1\omega_2 - \ldots = -\omega_1(x\xi);$$

further, as $X_{rr}X_{ss} - X_{rs}^2 = X\square$, where $\square$ is a certain minor of X of four rows and columns, it follows that X_{rs} also divides by $(x\xi)$. Thus every first minor of X divides by $(x\xi)$, identically in regard to (x).

(3) To express the tangent four-fold of the quadric, other than $(x\xi)$, passing through (A_1), (A_2), (A_3), (A_4), we introduce a sixth arbitrary point (A_0), not lying upon the linear four-fold determined by (A_1), (A_2), ..., (A_5). This four-fold must be capable of an equation of the form

$$\lambda(xA_0) + \lambda_1(xA_1) + \ldots + \lambda_5(xA_5) = 0,$$

and the conditions are that it should touch the quadric and pass through (A_1), (A_2), (A_3), (A_4). Putting

$$\Omega_0 = (A_0\xi) = \omega_1 c_{01} + \ldots + \omega_5 c_{05},$$

where

$$c_{0i} = (A_0A_i) = A_0A_i + B_0B_i + \ldots + H_0H_i,$$

we thus get for the equation required

$$(xA_5') = 0,$$

where $$(xA_5') = C_{50}(xA_0)+C_{51}(xA_1)+\ldots+C_{55}(xA_5)-\frac{C_{55}}{2\omega_5}(x\xi),$$

the quantities C_{ij} being first minors in the determinant of six rows and columns

$$C = \begin{vmatrix} c_{00}, & c_{01}, & \ldots, & c_{05} \\ \ldots & \ldots & \ldots & \ldots \\ c_{50}, & c_{51}, & \ldots, & c_{55} \end{vmatrix},$$

which is the same, with the substitution of (A_0) for (x), as that previously denoted by X. Thus all the first minors of C divide by Ω_0, and in particular $C_{i0} = -\omega_i\Omega_0$, while C itself is equal to $-\Omega_0^2$. It is easy to verify that (xA_5') is unaltered save for a constant factor when (A_0) is replaced by any other point independent of (A_1), ..., (A_5).

(4) From the expression for (xA_5), or in general for

$$(xA_i') \quad (i = 1, 2, \ldots, 5),$$

we find $$(A_0A_i') = -\frac{C_{ii}}{2\omega_i}\Omega_0, \quad (A_iA_j') = 0, \quad (A\ A_i') = C$$

$$(i, j = 1, \ldots, 5\ ;\ i \neq j),$$

and thence, if c_{ij}' denote $(A_i'A_j')$, we obtain

$$c_{ij}' = -C\,\frac{\omega_i^2 C_{jj}+\omega_j^2 C_{ii}-2\omega_i\omega_j C_{ij}}{2\omega_i\omega_j}.$$

We have remarked that X_{55} is a constant multiple of the product $(x\xi)(xA_5')$; hence we find

$$X_{55} = \frac{2\omega_5}{C}(x\xi)(xA_5'),$$

for this becomes an identity when (x) is replaced by (A_0). From this it follows that the value of X_{55} when (x) is at (A_1'), which we shall denote by $X_{55}^{(1)}$, is given by

$$X_{55}^{(1)} = 2\omega_5\omega_1 c_{15}' = -C(\omega_1^2 C_{55}+\omega_5^2 C_{11}-2\omega_5\omega_1 C_{51}) = X_1^{(5)},$$

being equal to the value of X_{11} when (x) is at (A_5').

Taking any symmetrical determinant of six rows and columns

$$A = |\,a_{00}, a_{01}, \ldots\,|,$$

considering the determinant B of five rows and columns obtained by erasing the first row and column in A, denoting the minors of the diagonal elements in B by $B_{11}, \ldots, B_{55}$, we obviously have the following identity, obtained by expanding the determinant of seven rows and columns obtained from A by bordering it with the elements $0, 0, \theta_1, \theta_2, \ldots, \theta_5$,

$$\begin{vmatrix} 0, & 0, & \theta_1, & \theta_2, & \ldots, & \theta_5 \\ 0, & a_{00}, & a_{01}, & a_{02}, & \ldots, & a_{05} \\ \theta_1, & a_{10}, & a_{11}, & a_{12}, & \ldots, & a_{15} \\ \ldots & & \ldots & \ldots & \ldots & \end{vmatrix}$$

$$= -(\theta_1\sqrt{B_{11}}+\ldots+\theta_5\sqrt{B_{55}})\left(\theta_1\frac{A_{11}}{\sqrt{B_1}}+\ldots+\theta_5\frac{A_{55}}{\sqrt{B_{55}}}\right)$$

$$+\Sigma\Sigma\theta_1\theta_2\frac{A_{11}B_{22}+A_{22}B_{11}-2A_{12}\sqrt{B_{11}}\sqrt{B_{22}}}{\sqrt{B_{11}}\sqrt{B_{22}}},$$

where A_{ij} are the minors in A of the elements a_{ij} respectively.

Suppose now that the determinant B, which is the minor of a_{00} in A, is zero, and that the signs of the square roots are determined so that

$$a_{11}\sqrt{B_{11}}+\ldots+a_{15}\sqrt{B_{55}} = 0;$$

by replacing $\theta_1, \theta_2, \ldots, \theta_5$ respectively by $a_{11}, a_{12}, \ldots, a_{15}$ and subtracting the third row from the first, and the third column from the first, these become

$$-a_{11}, \ -a_{10}, \ 0, \ 0, \ 0, \ 0, \ 0,$$

and the determinant is equal to $-a_{11}A-a_{10}^2B$ or $-a_{11}A$; if then we suppose $a_{11} = 0$, we have the identity

$$\Sigma\Sigma a_{11}a_{12}\frac{A_{11}B_{22}+A_{22}B_{11}-2A_{12}\sqrt{B_{11}}\sqrt{B_{22}}}{\sqrt{B_{11}}\sqrt{B_{22}}} = 0.$$

Take now the case of this in which the determinant A is our determinant C; we then infer the five identities

$$\sum_{r,s}^{1\ldots5}\sum c'_{rs}c_{ir}c_{is} = 0 \quad (i = 1, \ldots, 5).$$

(5) Now let a point (ξ') be defined by the identity in (x),

$$(x\xi') = \frac{C_{11}}{2\omega_1}(xA_1)+\ldots+\frac{C_{55}}{2\omega_5}(xA_5)-\Omega_0(xA_0);$$

it is easy to see by substitution that (ξ') is the point of intersection of the tangent four-folds of the quadric $(x^2) = 0$ at the points (A_1'), ..., (A_5').

The condition that this point (ξ') should lie on the quadric, or $(\xi'^2) = 0$ is, at once,

$$\Omega_0^2 c_{00} - \Omega_0 \left(c_{01} \frac{C_{11}}{\omega_1} + \ldots + c_{05} \frac{C_{55}}{\omega_5} \right) + \tfrac{1}{2} \sum\sum_{r,\,s} c_{r,s} \frac{C_{rr} C_{ss}}{\omega_r \omega_s} = 0.$$

This condition is necessarily independent of the arbitrary point (A_0), and if satisfied by any one position of (A_0), not lying on the planar four-fold containing (A_1), ..., (A_5), will be satisfied for any other position of (A_0); as we may verify directly by substituting $(A_0 + \theta_1 A_1 + \ldots + \theta_5 A_5)$ for (A_0). To find the condition that (ξ') lies on the quadric it is therefore sufficient to consider the case when (A_0) is replaced by (A_5').

Then, as $(A_5' A_i) = 0$, unless $i = 5$, and $(A_5'^2) = 0$, the condition reduces to

$$\sum\sum_{r,\,s} c_{rs} c'_{r5} c'_{s5} = 0;$$

and herein it will be noticed that terms for which r or s is equal to 5 disappear of themselves. This condition, as we have seen (4), p. 190, is the same as

$$\sum\sum_{r,\,s} c_{rs} X_{55}^{(r)} X_{55}^{(s)} / \omega_r \omega_s = 0,$$

where

$$X_{55} = \begin{vmatrix} (x^2), & (xA_1), & \ldots, & (xA_4) \\ (xA_1), & c_{11}, & \ldots, & c_{14} \\ \ldots & \ldots & \ldots & \ldots \\ (xA_4), & c_{41}, & \ldots, & c_{44} \end{vmatrix} \qquad (c_{ii} = 0,\ c_{ij} = c_{ji}),$$

and $X_{55}^{(r)}$ is the value of X_{55} when (x) is at (A_r'). If we take the determinant of four rows and columns

$$K = \begin{vmatrix} 0, & c_{12}, & c_{13}, & c_{14} \\ c_{21}, & 0, & c_{23}, & c_{24} \\ c_{31}, & c_{32}, & 0, & c_{34} \\ c_{41}, & c_{42}, & c_{43}, & 0 \end{vmatrix},$$

denoting by K_{rr} the minor of c_{rr} in this, and bear in mind that $(A_1'^2) = 0$, $(A_r A_r') = C$, $(A_s A_r') = 0$, when $s \neq r$ (p. 190), we see that

$$X_{55}^{(r)} = - C^2 K_{rr};$$

hence the equation in question reduces to

$$\sum_{r,s}^{1\ldots4}\sum c_{rs}\frac{K_{rr}K_{ss}}{\omega_r\omega_s}=0.$$

In this form the only reference to the quantities c_{15}, c_{25}, c_{35}, c_{45}, which do not occur in K, is in the quantities ω_1, ω_2, ω_3, ω_4, which satisfy the five equations

$$c_{i1}\omega_1+\ldots+c_{i5}\omega_5=0 \quad (i=1,\ldots,5),$$

and thence the equation

$$2\sum_{r,s}^{1\ldots4}\sum c_{rs}\omega_r\omega_s=\sum_1^4\omega_r(c_{r1}\omega_1+\ldots+c_{r4}\omega_4)=\sum_1^4\omega_r(-c_{r5}\omega_5)=-\omega_5\sum_1^4 c_{5r}\omega_r=0;$$

conversely, if ω_1, ω_2, ω_3, ω_4 be any four quantities whose ratios satisfy the equation

$$2\sum_{r,s}^{1\ldots4}\sum c_{rs}\omega_r\omega_s=0,$$

and we take ω_5 and c_{15}, c_{25}, c_{35}, c_{45}, so that, for $r=1, 2, 3, 4$,

$$\sum_{s=1}^{5} c_{rs}\omega_s=0,$$

these will give

$$\sum_{s=1}^{5} c_{5,s}\omega_s=0,$$

which are the original equations for $\omega_1, \ldots, \omega_5$, and lead to $C_{00}=0$.

6. The condition in question is thus, with a slight change of notation, reduced to this: if (ξ, η, ζ, τ) be any point of the quadric

$$Q\equiv f\eta\zeta+g\zeta\xi+h\xi\eta+f_1\xi\tau+g_1\eta\tau+h_1\zeta\tau=0,$$

and A, B, C, D denote the minors of the diagonal elements of the determinant

$$\begin{vmatrix} 0 & h & g & f_1 \\ h & 0 & f & g_1 \\ g & f & 0 & h_1 \\ f_1 & g_1 & h_1 & 0 \end{vmatrix}$$

does the point $\xi'=\frac{A}{\xi}$, $\eta'=\frac{B}{\eta}$, $\zeta'=\frac{C}{\zeta}$, $\tau'=\frac{D}{\tau}$,

also lie upon the quadric Q? This is immediately answered in the affirmative, but it is to be noted that this is the first fact we utilise requiring the determinants to be of the precise number of rows and columns

which we have taken. The preceding work applies with slight changes to points upon a quadric $(x^2) = 0$ in any number of dimensions. For this determinant of four rows and columns

$$A = 2fg_1h_1, \quad B = 2gh_1f_1, \quad C = 2hf_1g_1, \quad D = 2fgh,$$

$$f\eta'\zeta' = \frac{4fgh f_1g_1h_1}{\xi\eta\zeta\tau} f_1\xi\tau = \sigma f_1\xi\tau, \quad f_1\xi'\tau' = \sigma f\eta\zeta, \quad \ldots,$$

and the identity required is obvious. Similarly the existence of a double $(n+1)$ of points upon a quadric in n dimensions requires a theorem of self-transformation for a quadric in $(n-2)$-dimensions.

For the case of five dimensions another analytical proof, much briefer than the preceding, professedly depending on an ingenious manipulation of the algebra is given by Mr. Richmond, *Camb. Phil. Proc.*, Vol. XIV, p. 477 (Feb. 25th, 1908); and another, depending upon the properties of quadric surfaces in three dimensions, by Prof. A. C. Dixon, *Quar. Jour.*, Vol. XL, 1909, pp. 381–384. Mr. Richmond remarks, as an analytic equivalent of the theorem of the existence of a double six, that "if in a symmetrical determinant of six rows and columns, the six elements in the leading diagonal all vanish, and the first minors of five of these elements also vanish, the minor of the remaining diagonal element must also vanish." It may be remarked that the complementary theorem, that "if in a symmetrical determinant of six rows and columns all the six principal minors vanish, and also all the six principal elements vanish except one, that remaining element must also vanish" is not true. For consider, in the notation employed above, the determinant

$$X = \begin{vmatrix} (x^2), & (xA_1), & \ldots, & (xA_5) \\ (xA_1), & c_{11}, & \ldots, & c_{15} \\ \ldots & \ldots & \ldots & \ldots \\ (xA_5), & c_{51}, & \ldots, & c_{55} \end{vmatrix}.$$

If π denote the tangent four-fold of the quadric $(x^2) = 0$ at (ξ), and π_i' the tangent four-fold at (A_i'), we have seen that the first minor X_{ii} vanishes for any point (x) either on π or π_i'. Thus $X_{11}, X_{22}, \ldots, X_{55}$ all vanish for any point on the one-fold (or line) joining (A_1) and (A_2), both of which lie on π_3', π_4', and π_5'. And the remaining principal minor of X, that of (x^2), also vanishes, as do all the principal elements $c_{11}, c_{22}, \ldots, c_{55}$. But the remaining principal element (x^2) does not vanish for every point on A_1A_2.

(7) With the formulæ now obtained many interesting problems may be worked out. We may put down the equation in line coordinates of the quadric surface in regard to which the six pairs (A_1), (A_1'), ..., (ξ), (ξ') are polar reciprocals; or we may determine the coordinates of the remaining fifteen lines of the cubic surface constructed to have these six pairs as a double six; the figure in five dimensions brings out clearly the desirability of considering, as derived from the two pairs (A_1), (A_1') and (A_2), (A_2'), not only the line lying in the two planes (in three dimensions) determined by (A_1, A_2') and by (A_2), (A_1'), but also the line joining the points (in three dimensions) determined by (A_1), (A_2') and by (A_2), (A_1'). The former lies on the cubic surface referred to, the latter does not. We refrain from these questions here. But it seems worth while to refer briefly to the reciprocity which necessarily exists between the two sets (ξ), (A_1'), ..., (A_5') and (ξ'), (A_1), ..., (A_5). Putting P, P_1 for the two matrices of six rows and columns

$$P = \begin{Bmatrix} \xi, & \eta, & \dots \\ A_1', & B_1', & \dots \\ \dots & \dots & \dots \\ A_5', & B_5', & \dots \end{Bmatrix}, \qquad P_1 = \begin{Bmatrix} \xi', & \eta', & \dots \\ A_1, & B_1, & \dots \\ \dots & \dots & \dots \\ A_5, & B_5, & \dots \end{Bmatrix},$$

we have, in virtue of the relations given above (4), p. 190, &c.,

$$P\bar{P}_1 = \begin{Bmatrix} C & 0 & 0 & \dots \\ 0 & C & 0 & \dots \\ \dots & \dots & \dots & \dots \end{Bmatrix} = C;$$

this leads to $$\bar{P}_1 P = P_1 \bar{P} = \bar{P} P_1 = C,$$

and hence $$P\bar{P}P_1\bar{P}_1 = C^2,$$

namely,

$$\begin{Bmatrix} 0 & (\xi A_1') & (\xi A_2') & \dots \\ (\xi A_1') & 0 & c_{12}' & \dots \\ (\xi A_2') & c_{21}' & 0 & \dots \\ \dots & \dots & \dots & \dots \end{Bmatrix} \begin{Bmatrix} 0 & (\xi' A_1) & (\xi' A_2) & \dots \\ (\xi' A_1) & 0 & c_{12} & \dots \\ (\xi' A_2) & c_{21} & 0 & \dots \\ \dots & \dots & \dots & \dots \end{Bmatrix} = C^2.$$

(8) The consideration of the correspondence between the space of five dimensions and the original space of three dimensions leads to interesting results, of which we give some examples.

If in the conditions that a line of coordinates (l, m, n, l', m', n') should contain a point (a, b, c), namely,

$$l'+mc-nb=0, \quad \ldots, \quad l'a+m'b+n'c=0,$$

we put
$$l=x+u, \quad m=y+v, \quad n=z+w,$$
$$l'=x-u, \quad m'=y-v, \quad n'=z-w,$$

we find
$$(u, v, w)=(1+\gamma)(1-\gamma)^{-1}(x, y, z),$$

where γ is the matrix
$$\gamma=\begin{Bmatrix} 0, & -c, & b \\ c, & 0, & -a \\ -b, & a, & 0 \end{Bmatrix},$$

namely the familiar expressions, known as Rodrigues', for a positive orthogonal transformation, that is an orthogonal transformation whose determinant is $+1$. (Cf. Rodrigues, *Liouville's Journal*, V; Cayley, *Collected Papers*, I, p. 29; I, p. 124; II, p. 505.) Similarly the conditions that the line (l, m, n, l', m', n') should lie in the plane

$$aX+bY+cZ+1=0,$$

are
$$(u, v, w)=-(1+\gamma)(1-\gamma)^{-1}(x, y, z),$$

being those of a negative orthogonal transformation. If we interpret (x, y, z, u, v, w) as the homogeneous coordinates of a point of five dimensions, any line of our original space S_3, corresponds to a point in S_5 lying upon the quadric
$$u^2+v^2+w^2=x^2+y^2+z^2,$$

intersecting lines corresponding to conjugate points of this quadric (each lying in the tangent four-fold at the other). The lines of S_3 through a given point (a, b, c) correspond then to the points of S_5 which lie on the two-fold expressed by

$$(u, v, w)=\begin{Bmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{Bmatrix}(x, y, z)=\Delta(x, y, z), \text{ say,}$$

where
$$\Delta=(1+\gamma)(1-\gamma)^{-1},$$

these equations involving that also

$$u^2+v^2+w^2=x^2+y^2+z^2;$$

the converse expressions of a, b, c in terms of l_1, m_1, ..., n_3, are

$$a = \frac{n_1+l_3}{l_2-m_1} = \frac{l_2+m_1}{n_1-l_3}, \quad b = \frac{l_2+m_1}{m_3-n_2} = \frac{m_3+n_2}{l_2-m_1}, \quad c = \frac{m_3+n_2}{n_1-l_3} = \frac{n_1+l_3}{m_3-n_2},$$

with $$l_1 = \frac{n_1 l_2 - m_3}{n_2} = \frac{l_3 m_1 - n_2}{m_3}, \quad m_2 = \frac{m_1 n_2 - l_3}{n_1} = \frac{l_2 m_3 - n_1}{l_3},$$

$$n_3 = \frac{n_2 l_3 - m_1}{l_2} = \frac{m_3 n_1 - l_2}{m_1}.$$

There exist in all ∞^3 two-folds $(u, v, w) = \Delta(x, y, z)$ of the above form; there being ∞^1 of these through any point $(x_0, \ldots, w_0)$ of the quadric, all lying on the tangent four-fold, $uu_0+\ldots = xx_0+\ldots$ at the point; through any point of this tangent four-fold which lies on the quadric and the point $(x_0, \ldots, w_0)$ itself passes a definite one of these two-folds. It seems desirable, for clearness sake, to have a definite name for these two-folds $(u, v, w) = \Delta(x, y, z)$; we shall call them *generators*, and in fact positive generators, the determinant of Δ being $+1$. There exist, however, negative generators $(u, v, w) = \Delta'(x, y, z)$, Δ' being a matrix of determinant -1, in regard to which similar remarks may be made. An obvious way in which to pass from a positive generator to a negative generator is to change the sign of every element in any one row of Δ, or in any one column; by all possible combinations of these processes any generator gives rise, including itself, to thirty-two generators, sixteen of them positive and sixteen negative, any negative generator arising from a particular positive generator by change of sign of every element of this. If we consider the sixteen points (a, b, c) in S_3, corresponding to the sixteen generators obtainable from a positive generator by change of sign of an even number of rows and columns in the matrix of transformation Δ, we find (using homogeneous coordinates) that they are such points as $(a, b, c, 1)$, $(-b, a, 1, -c)$, $(-b, -a, 1, c)$, ..., forming a Kummer configuration, that is, such as lie by sixes on sixteen planes, six through each point, the six points in any plane lying on a conic, and the six planes through any point touching a quadric cone. The six points lying in the plane $aX+bY+cZ+d = 0$ correspond, in fact, to the matrices obtained from Δ, by changing the signs of the elements of any one of the three sets of two rows, or of any one of the three sets of two columns.

The points of a positive generator

$$u = l_1 x + m_1 y + n_1 z, \quad v = l_2 x + m_2 y + n_2 z, \quad w = l_3 x + m_3 y + n_3 z,$$

or

$$(u, v, w) = \Delta(x, y, z),$$

which lie in a four-fold

$$Ax+By+Cz+Fu+Gv+Hw=0,$$

lie also in a negative generator

$$(u,\ v,\ w) = \Delta'(x,\ y,\ z),$$

given by
$$\Delta' = \Delta + \begin{Vmatrix} \sigma_1 p, & \sigma_1 q, & \sigma_1 r \\ \sigma_2 p, & \sigma_2 q, & \sigma_2 r \\ \sigma_3 p, & \sigma_3 q, & \sigma_3 r \end{Vmatrix},$$

where $p = A+Fl_1+Gl_2+Hl_3, \quad q = B+Fm_1+Gm_2+Hm_3,$

$$r = C+Fn_1+Gn_2+Hn_3,$$

$$\sigma_1 = -2(pl_1+qm_1+rn_1)/(p^2+q^2+r^2), \quad \ldots .$$

If we consider six four-folds

$$A_r x+B_r y+C_r z+F_r u+G_r v+H_r w=0 \quad (r=1,\ \ldots,\ 6),$$

such that for any two of them we have the condition

$$A_r A_s+B_r B_s+C_r C_s-(F_r F_s+G_r G_s+H_r H_s)=0 \quad (r,\ s=1,\ \ldots,\ 6),$$

and, as we may without loss of generality, we impose the further conditions

$$A_r^2+B_r^2+C_r^2-(F_r^2+G_r^2+H_r^2)=1 \quad (r=1,\ \ldots,\ 6),$$

the matrix of six rows and columns of which one row is

$$(A_r,\ B_r,\ C_r,\ iF_r,\ iG_r,\ iH_r)$$

is orthogonal, and we hence infer that also

$$A_1^2+A_2^2+\ldots+A_6^2=1,\ \ldots,\ F_1^2+\ldots+F_6^2=-1,\ \ldots,$$

$$B_1C_1+\ldots+B_6C_6=0,\ \ldots,\ A_1G_1+\ldots+A_6G_6=0,\ \ldots .$$

Now consider the six one-fold (or line) intersections of these six four-folds all with the same positive generator

$$(u,\ v,\ w) = \begin{Vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{Vmatrix} (x,\ y,\ z);$$

these satisfy respectively the equations $(r=1,\ 2,\ \ldots,\ 6)$

$$P_r \equiv (A_r+F_r l_1+G_r l_2+H_r l_3)\,x+(B_r+F_r m_1+G_r m_2+H_r m_3)\,y$$
$$+(C_r+F_r n_1+G_r n_2+H_r n_3)\,z=0,$$

which we may look upon as the equations referred to a triangle of reference in the (two-fold) generator $(u, v, w) = \Delta(x, y, z)$ of the (lines or) one-fold intersections in question. The expression

$$P_1^2 + \ldots + P_6^2,$$

in virtue of the orthogonal relations above, is equal to

$$x^2(1 - l_2^1 - l_2^2 - l_3^2) + \ldots + 2yz(-m_1 n_1 - m_2 n_2 - m_3 n_3) + \ldots,$$

that is identically to zero. The six lines of intersection therefore touch a conic. In the space S_3 this is the proposition that the planes of six linear complexes, mutually in involution, which pass through an arbitrary point, touch a quadric cone.

RELATIONS BETWEEN ARITHMETICAL BINARY CUBIC FORMS AND THEIR HESSIANS

By G. B. MATHEWS.

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1. The principal object of this paper is to show how to find all the arithmetical cubics which have a Hessian of the form $\mu(Ax^2+Bxy+Cy^2)$, where A, B, C are assigned integers and μ is an integral multiplier. The result is that every such cubic can be expressed in the form

$$f = m\phi+n\psi,$$

where ϕ, ψ are determinate cubics and m, n are integers. By giving to m, n all possible integral values, we obtain all the cubics associated with $Ax^2+Bxy+Cy^2$, and the multiplier μ is a determinate quadratic form in m, n.

2. The algebraic theory enables us to see that, putting

$$B^2-4AC = D,$$

we must have

$$f = l\{2Ax+(B+\sqrt{D})y\}^3+m\{2Ax+(B-\sqrt{D})y\}^3,$$

where l, m are numerical; and we find, as the Hessian of f,

$$\bar{h}(f) = 64lmA^3D(Ax^2+Bxy+Cy^2),$$

where the bar over the h indicates that f has been treated as a form with binomial coefficients attached to it.

Now, in order that f may have rational coefficients, we must have

$$l = p+q\sqrt{D}, \qquad m = p-q\sqrt{D},$$

where p, q are rational; and these values give

$$\bar{h}(f) = 64(p^2-Dq^2)A^3D(Ax^2+Bxy+Cy^2).$$

3. With this notation

$$f = ax^3+3bx^2y+3cxy^2+dy^3,$$

where
$$a = 16pA^3,$$
$$b = 8pA^2B+8qA^2D,$$
$$c = 8pA(B^2-2AC)+8qABD,$$
$$d = 8pB(B^2-3AC)+8q(B^2-AC)D.$$

Finally, then, if we write

$$8p = t, \qquad 8qD = s,$$

we shall have
$$f = \alpha x^3+\beta x^2y+\gamma xy^2+\delta y^3,$$

with
$$h(f) = 9(Dt^2-s^2)A^3(Ax^2+Bxy+Cy^2),$$

and
$$\left.\begin{aligned} \alpha &= 2tA^3 \\ \beta &= 3tA^2B+3sA^2 \\ \gamma &= 3tA(B^2-2AC)+3sAB \\ \delta &= tB(B^2-3AC)+s(B^2-AC) \end{aligned}\right\}. \tag{1}$$

4. We shall obtain the most general form of f by finding the most general forms of s, t, which make α, β, γ, δ integral. From the first two of the equations (1),

$$t = \frac{\alpha}{2A^3}, \qquad s = \frac{-3B\alpha+2A\beta}{6A^3},$$

and substituting these values in the others, we find

$$\gamma = \frac{-3C\alpha+B\beta}{A}, \qquad \delta = \frac{-3BC\alpha+(B^2-AC)\beta}{3A^2}. \tag{2}$$

Therefore the necessary and sufficient conditions which have to be satisfied are

$$\left.\begin{aligned} -3C\alpha+B\beta &\equiv 0 \pmod{A} \\ -3BC\alpha+(B^2-AC)\beta &\equiv 0 \pmod{3A^2} \end{aligned}\right\}. \tag{3}$$

These are congruences which have a solution of the form

$$\alpha = m\alpha_1+n\alpha_2, \qquad \beta = m\beta_1+n\beta_2,$$

where m, n are arbitrary integers, and from each such solution we obtain

$$\gamma = m\gamma_1+n\gamma_2, \qquad \delta = m\delta_1+n\delta_2,$$

leading to $f = m\phi+n\psi$, as stated in the introduction.

5. So far nothing has been assumed with regard to A, B, C, except that they are integers; but the most interesting cases are where they

have no common factor, and when D is positive and not a square. Moreover, we may assume that $Ax^2+Bxy+Cy^2$ is a reduced form.

As an example, let

$$A, B, C = 3, 3, -8, \qquad D = 105.$$

The congruences to be satisfied are

$$24\alpha+3\beta \equiv 0 \pmod 3,$$

$$72\alpha+33\beta \equiv 0 \pmod{27}.$$

The first is identical, the second reduces to

$$24\alpha+11\beta \equiv 0 \pmod 9,$$

whence $\beta = 3y$, and $\quad 8\alpha+11y \equiv 0 \pmod 3,$

or $\quad \alpha+y \equiv 0 \pmod 3.$

Putting $\quad \alpha = 3m-n, \qquad \beta = 3n,$

we have $\quad \gamma = 24m-5n, \qquad \delta = 8m+n;$

$$f = m(3x^3+24xy^2+8y^3)+n(-x^3+3x^2y-5xy^2+y^3),$$

$$h(f) = (72m^2-39mn+2n^2)(3x^2+3xy-8y^2).$$

It will be observed that the multiplier here is a quadratic form in m, n of determinant 945, that is 9.105. We may, if we like, reduce this by the proper unitary substitution: in this case it is

$$m = n_1,$$

$$n = -m_1+3n_1,$$

leading to $\quad f = m_1(x^3-3x^2y+5xy^2-y^3)+n_1(9x^2y+9xy^2+11y^3),$

with $\quad h(f) = (2m_1^2+27m_1n_1-27n_1^2)(3x^2+3xy-8y^2).$

Other examples are

$D = 5,$ $\quad f = m(x^3+3x^2y+y^3)+n(3x^2y+3xy^2+2y^3),$

$\quad h = 9(m^2+mn-n^2)(x^2+xy-y^2);$

$D = 13,$ $\quad f = m(x^3+9xy^2+3y^3)+n(3x^2y+3xy^2+4y^3),$

$\quad h = 9(3m^2+mn-n^2)(x^2+xy-3y^2);$

$D = 20,$ $\quad f = m(x^3+15xy^2)+n(3x^2y+5y^3),$

$\quad h = 9(5m^2-n^2)(x^2-5y^2);$

$D = -3,$ $\quad f = m(x^3-3xy^2-y^3)+n(x^2y+xy^2),$

$\quad h = (-9m^2+3mn-n^2)(x^2+xy+y^2).$

6. There are two interesting special cases: the first is when

$$A, B, C = 1, 0, -N, \quad D = 4N.$$

The congruences to be solved are

$$3N\alpha \equiv 0 \pmod 1,$$
$$N\beta \equiv 0 \pmod 3.$$

Hence, if N is prime to 3,

$$\alpha = m, \quad \beta = 3n, \quad \gamma = 3Nm, \quad \delta = Nn,$$
$$f = m(x^3+3Nxy^2)+n(3x^2y+Ny^3),$$

with
$$h = 9(Nm^2-n^2)(x^2-Ny^2).$$

On the other hand, if $N = 3N'$,

$$\alpha = m, \quad \beta = n, \quad \gamma = 9N'm, \quad \delta = N'n,$$
$$f = m(x^3+9N'xy^2)+n(x^2y+N'y^3),$$
$$h = (27N'm^2-n^2)(x^2-3N'y^2).$$

The other case is when

$$A, B, C = 1, 1, -N, \quad D = 4N+1.$$

Here the congruences are $3N\alpha+\beta \equiv 0 \pmod 1,$

$$3N\alpha+(N+1)\beta \equiv 0 \pmod 3,$$

and there are two alternatives.

First, suppose $(N+1)$ prime to 3: then

$$\alpha = m, \quad \beta = 3n, \quad \gamma = 3Nm+3n, \quad \delta = Nm+(N+1)n;$$
$$f = m(x^3+3Nxy^2+Ny^3)+n\{3x^2y+3xy^2+(N+1)y^3\},$$
$$h = 9(Nm^2+mn-n^2)(x^2+xy-Ny^2).$$

Secondly, suppose $N = 3N'-1$; then

$$\alpha = m, \quad \beta = n, \quad \gamma = (9N'-3)m+n, \quad \delta = (3N'-1)m+N'n;$$
$$f = m\{x^3+(9N'-3)xy^2+(3N'-1)y^3\}+n\{x^2y+xy^2+N'y^3\},$$
$$h = \{(27N'-9)m^2+3mn-n^2\}\{x^2+xy-(3N'-1)y^2\}.$$

7. Consider again the first case of Art. 6. Let (T, U) be the fundamental solution of

$$T^2-NU^2 = 1,$$

and let the others be defined by

$$t_n+u_n\sqrt{N} = (T+U\sqrt{N})^n.$$

It follows from what has been said that, if

$$f = U(x^3+3Nxy^2)+T(3x^2y+Ny^3),$$

then $$h = -9(x^2-Ny^2).$$

Now the automorphs of h are given by

$$x = t_n x_n+Nu_n y_n,$$

$$y = u_n x_n+t_n y_n,$$

and if this be applied to f, the result is found to be

$$f_n = u_{3n+1}(x^3+3Nxy^2)+t_{3n+1}(3x^2y+Ny^3),$$

so we have a sequence of equivalent cubics $f_1, f_2, \ldots$, all having the Hessian $-9(x^2-Ny^2)$.

The cubicovariant of f_n, defined as the Jacobian of f_n and h, is found to be

$$q_n = 54\{t_{3n+1}(x^3+3Nxy^3)+Nu_{3n+1}(3xy^2+Ny^3)\},$$

and its Hessian is $$54^2N(x^2-Ny^2).$$

Similar results are obtained for any other given Hessian and its associated cubics, but they are of a less simple character.

In conclusion it may be observed that the method of this paper gives the complete solution of the Diophantine equations

$$3\alpha\gamma-\beta^2 = \mu A, \quad 9\alpha\delta-\beta\gamma = \mu B, \quad 3\beta\delta-\gamma^2 = \mu C,$$

where A, B, C are given integers, and α, β, γ, δ, μ have to be found.

ON GEISER'S METHOD OF GENERATING A PLANE QUARTIC

By Marjorie Long.

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1.

I propose in this paper to prove some of the properties of the bitangents of a plane quartic curve by Geiser's method of projecting a cubic surface from a point of itself. The properties which I shall discuss come under three main divisions, according as they belong to groups of bitangents (*a*) whose points of contact lie on a conic, (*b*) which touch a conic, or (*c*) which intersect by pairs in collinear points; but it will be seen that these properties overlap. In the case of (*b*) and (*c*), the results obtained lead to some interesting properties of the cubic surface,* which are probably some of those referred to in Prof. Cayley's remark, quoted by Mr. Richmond,† that the "remarkable fact" that "any tangent plane of a cubic surface and the twenty-seven planes through the point of contact and the twenty-seven lines of the surface form an absolutely symmetrical set of planes, must imply a series of quite unknown properties of the cubic surface."

The properties of the bitangents in question have already been fully discussed from other points of view. A useful summary of results (with references to earlier memoirs) has been given by Timerding,‡ and I am

* See §§ 13 and 29.

† *Quarterly Journal*, Vol. xxxi, p. 160.

‡ *Crelle's Journal*, Vol. cxxii (1900), p. 209.

indebted to this paper for several suggestions and for some of the terms used in describing the grouping of the bitangents. Geiser's method has the advantage of giving an elementary proof by synthetic methods of theorems which otherwise require long algebraic analysis. By its use the number and mutual relations of groups of bitangents of various kinds become almost intuitive, so that it seems worth while to examine some of the cases in more detail than has been done up to the present. I have also, in most cases, been able to give two or three independent methods of determining accurately the numbers concerned. A summary of references to Geiser's method will be found at the end of the paper.

The greater part of the contents of this paper has been discussed with Dr. H. F. Baker, Fellow of St. John's College, Cambridge, to whom I owe much valuable criticism and many suggestions.

I have assumed the existence and most obvious properties of twenty-seven straight lines on a general cubic surface. They are discussed fully by Sturm ("Flächen dritter Ordnung"), Cremona (*Crelle's Journal*, Bd. LXVIII), and Salmon ("Geometry of Three Dimensions"). I give here (§ 2) a summary of as much of Geiser's work as seems necessary for my purpose. For further details reference may be made to the original memoir,* "Ueber die Doppeltangenten einer ebenen Curve vierten Grades."

2.

The tangent cone to a general cubic surface F_3 without singularities, the vertex being a point P on F_3, is of the fourth degree. A plane through P and one of the lines g on the surface meets the surface in g, and a conic section which meets g in two points. The tangent cone touches the plane (P, g) along the lines joining these two points to P; and the tangent plane at P to F_3 touches the cone along the two inflexional tangents at P; so that the cone has the twenty-seven planes (P, g) and the tangent plane at P for double tangent planes. Hence, if we take the section of the cone by any plane, we shall get a plane quartic curve C_4; the twenty-seven lines on F_3 project into twenty-seven bitangents of the curve, and the intersection with the plane of C_4 of the tangent plane at P to F_3 is the twenty-eighth.

Conversely, any plane quartic curve can be regarded as the section of a tangent cone of some cubic surface. Any point P outside the plane of the cubic can be chosen for the vertex of the cone, and any one of the bitangents can be chosen to correspond to the tangent plane at P to the

* *Math. Annalen*, Bd. I (1869), p. 129.

cubic surface. In fact, if we take the selected bitangent as axis of z, and make the x- and y-axes go through its points of contact, the equation of the quartic will be of the form

$$z\phi_3(x, y, z) - x^2y^2 = 0,$$

where ϕ_3 is homogeneous and of the third degree in x, y, z. If we choose a point P outside this plane, and take as coordinate planes the planes which join P to the axes of x, y, z, then the surface whose equation is

$$F_3 \equiv \phi_3(x, y, z) + 4uxy + 4u^2z = 0$$

is the one required, since the tangent cone to it from P cuts the u-plane in the given quartic.

3.

The cubic surfaces considered in this paper are in all cases perfectly general and without singularities. If the surface is specialised in any way (*e.g.*, by the presence of a conical point, or by two or more lines on it coalescing), then any plane quartic derived from it will have some corresponding singularity; and the converse is also true. The forms of the cubic surface and the associated quartics when some of the lines become imaginary have been discussed by Zeuthen,* and the case of a conical point has been examined by Cremona in a paper on "Pascal's Hexagram." †

It may happen that some of the bitangents have imaginary points of contact, *e.g.*, the bitangent which corresponds to the tangent plane at the point of projection P will touch the quartic in real points only if the inflexional tangents at P are real, *i.e.*, if P is a point of hyperbolic curvature. But, since all the properties under discussion are projective and independent of the reality of the elements involved, I have not thought it necessary to make any distinction between the cases.

4.

I denote the twenty-eighth bitangent, corresponding to the tangent plane at the point of projection, by L. The notation for the other twenty-seven bitangents will be introduced as required. Two lines in space which intersect are *plane*, two which do not are *skew*. Five skew lines which have a single common transversal I call *A lines*, five which have two common transversals I call *B lines*. Three or more bitangents whose

* *Math. Annalen*, Bd. VIII.

† *Atti della R. Accad. dei Linc.*, 1877.

points of contact with the quartic lie on a conic are *syzygetic*, and in the contrary case they are *azygetic* (these terms are due to Frobenius*). When no ambiguity is possible, a line on F_3 and the bitangent which is its projection will be denoted by the same letter or symbol.

5.

The plane through P and any line g on F_3 cuts F_3 in a conic which passes through P and meets g in two points. These points lie on the first polar of P with respect to F_3, and the tangent cone from P touches the plane (P, g) along the lines joining P to them. Hence these lines meet the plane of the quartic at the points of contact of the bitangent which is the projection of g. The two inflexional tangents $t_1 t_2$ of F_3 at P lie wholly on the first polar of P, and the tangent plane at P touches the tangent cone along t_1 and t_2. Hence t_1, t_2 meet the plane of the quartic in the points of contact of L.

Now consider three plane lines on F_3 and the points $\tau_1\tau_2$ in which their plane meets $t_1 t_2$. The plane through P and any one of the lines meets F_3 in a conic, which has two points common with the line. We thus get six points which, with $\tau_1\tau_2$, all lie on the first polar of P, which is of the second degree. The eight points also lie in a plane, and therefore they must lie on a conic.

Projecting into the quartic C_4, the points of contact of L and three bitangents, which are the projections of three plane lines, lie on a conic, *i.e.*, the four bitangents are syzygetic.

Now take the bitangents, which are the projections of two plane pairs of lines, $a_1 a_2$, $b_1 b_2$ which meet another line c. Let the points of contact of a_1 be $a_1 a_1'$, ..., and of L, $\lambda\lambda'$. Then $\lambda\lambda'\gamma\gamma' a_1 a_1' a_2 a_2'$ lie on a conic k_1 and $\lambda\lambda'\gamma\gamma'\beta_1\beta_1'\beta_2\beta_2'$ lie on a conic k_2. The quartic C_4 and the degenerate quartic $k_1 k_2$ are two of the pencil of curves of the fourth order through the four points $\lambda\lambda'\gamma\gamma'$ each taken twice, and the eight points α and β. The eight points $\lambda\lambda'\gamma\gamma'$ lie on the conic Lc, and it follows by a well known theorem† that the eight points α and β lie also on a conic. Thus the four bitangents $a_1 a_2 b_1 b_2$ are syzygetic.

Since we obtain a syzygetic group by projecting (a) two of the plane pairs of lines which meet one of the twenty-seven, or (b) three plane lines and L, it follows that we get in this way $27\times5\times4\div2+45 = 315$ syzygetic fours. Any three bitangents belonging to the same four are

* *Crelle's Journal*, Bd. XCIX.

† Cf. Geiser, *Crelle's Journal*, Bd. LXXII, p. 372.

syzygetic, so that we get $315 \times 4 = 1260$ syzygetic triplets, and the following groups of lines on F_3 project into syzygetic triplets :—

(*a*) Three plane lines.

(*b*) Two plane lines and one which is skew to both.

(*c*) Two plane lines and L.

It will be shown in § 9 that there cannot be any more syzygetic triplets, and so all other combinations of three lines on F_3 project into azygetic triplets, viz. :—

(*a*) Three skew lines.

(*b*) Two skew lines and one which meets both.

(*c*) Two skew lines and L.

6.

If three bitangents are syzygetic, there is a fourth which is syzygetic with them. For, if the three bitangents are a, b, c, and $aa'\beta\beta'\gamma\gamma'$ are their points of contact, so that $aa'\beta\beta'\gamma\gamma'$ lie on a conic k, then k will cut C_4 in two more points $\delta\delta'$. The conic k taken twice and C_4 are two of the pencil of quartics through the eight points $aa'\beta\beta'\gamma\gamma'\delta\delta'$ each taken twice. The eight points $a\beta$ lie on the conic ab, and therefore the other eight points $\gamma\delta$ lie on a conic touching C_4 at $\gamma\gamma'\delta\delta'$. This conic contains the line c as part of itself, and therefore another straight line must touch C_4 at $\delta\delta'$, *i.e.*, the line joining $\delta\delta'$ is a bitangent which is syzygetic with the first three.

7.

Consider the groups of six pairs of lines* :

(*a*) The five plane pairs of lines which meet one line on F_3, this line and L. I call this configuration of lines a *two-ten*.

(*b*) The six pairs of a double six on F_3.

We get, by projecting either group, six pairs of bitangents such that any two pairs are syzygetic ; and it is impossible to find any other arrangement of lines on F_3 which have this property. Such a group of twelve bitangents is called a *Steiner complex*. We thus get $27+36 = 63$ Steiner complexes.

This number can be found in other ways. If we define a Steiner complex as the projection of any line on F_3 and the ten lines which meet

* Geiser, *Crelle's Journal*, Bd. lxxii, p. 376.

it, with L, we get twenty-seven complexes from the cubic surface corresponding to a given L. By taking each bitangent in turn as L, we get apparently 27×28 complexes. But any one of these will occur when any one of the twelve lines contained in it is chosen as L, so that we get in reality $27 \times 28 \div 12 = 63$ Steiner complexes.

Similarly, if we define a Steiner complex as the projection of a double six, we get thirty-six complexes corresponding to a given L. But any one will occur when any of the sixteen lines which do not belong to it is chosen as L; and we get, as before, $28 \times 36 \div 16 = 63$ Steiner complexes.

8.

From the twenty-eight bitangents we can make $28 \times 27 \div 2 = 378$ pairs. These fall into sixty-three groups of six pairs each, each group being a Steiner complex. When any pair is chosen, the other five pairs which complete the Steiner complex to which it belongs are uniquely determined, *i.e.*, a given pair occurs in only a single Steiner complex. Consider the different cases on F_3. A pair of bitangents is represented by :—

(a) Two skew lines. These completely determine the double six of which they are a pair, and they cannot be a pair of a two-ten.

(b) Two plane lines. These cannot be a pair of a double six. The third line in their plane, the other eight lines which meet it, and L, complete the two-ten of which the two are a pair.

(c) L and a line g on F_3. These are a pair of the two-ten which is completed by the five plane pairs of lines which meet g.

In each case there is no possible ambiguity, and a unique Steiner complex is completely determined.

9.

We can now shew that no triplet can be syzygetic except those already enumerated. Let (a, b, c) be a triplet which has not already been proved to be syzygetic. If we suppose that it is, we have seen (§ 6) that there will be a fourth bitangent, d, syzygetic with a, b, c. Consider the Steiner complex of which (ab) is a pair. By hypothesis, c and d do not belong to this complex, and therefore we have *six* pairs of lines which are syzygetic with ab. We have seen that, if two pairs of lines are syzygetic with a third pair, they are syzygetic with each other (§ 5), and therefore cd

are syzygetic with every other pair of the Steiner complex, so that we have now *seven* pairs of lines such that any two pairs are syzygetic.

Now, if we take two bitangents as axes of x and y, the equation of the quartic becomes*

$$xyU = V^2,$$

where $U = 0$, $V = 0$ are the equations of conics. This may also be written

$$xy(\lambda^2 U + 2\lambda V + xy) = (xy + \lambda V)^2.$$

The discriminant of $\lambda^2 U + 2\lambda V + xy$ is of the sixth degree in λ, so that, besides the value $\lambda = 0$, which corresponds to the pair of lines xy, there are five values of λ for which $\lambda^2 U + 2\lambda V + xy = 0$ represents a pair of straight lines. If w and z are such a pair, so that the equation of the quartic is

$$xyzw = V'^2,$$

it is clear that the points of contact of each of the lines $x\,y\,z\,w$ lie on the conic $V' = 0$. Thus there are five, and only five, pairs of bitangents which are syzygetic with a given pair. Hence the supposition that $a\,b\,c$ are syzygetic is erroneous, and all the syzygetic triplets are exhausted by the enumeration given above.

10.

Four syzygetic lines can be arranged in pairs in three ways, and each arrangement will determine a Steiner complex. We have two cases to consider on F_3:

(*a*) The four lines are $a_1\,a_2\,b_1\,b_2$, of which the pairs $(a_1\,b_2)$, $(a_2\,b_1)$ are plane, but all other pairs are skew. The pair $(a_1\,b_1)$ determine a double six, of which $(a_2\,b_2)$ are another pair; the remaining eight lines meet either a_1 and a_2 or b_1 and b_2. The pairs $(a_1\,a_2)$, $(b_1\,b_2)$ determine another double six; the remaining eight lines meet either a_1 and b_1 or a_2 and b_2. Since the only line which can meet more than two of $a_1\,a_2\,b_1\,b_2$ is the line c, which lies in both the planes $(a_1\,b_2)$, $(a_2\,b_1)$, and so meets all four, and c does not belong to either double six, the sixteen lines so obtained are all distinct. There are seven lines left, which are c, and the other three plane pairs which meet c. These seven lines, with the original four and L, make up a two-ten.

(*b*) The four lines are L and three plane lines a, b, c. L and any one (*a*) of these three determine a two-ten, which contains the other four pairs of lines which meet a. The two-ten of which (L, b) is a pair contains the

* Salmon, *Higher Plane Curves*, § 255; Weber, *Algebra*, Bd. II, § 113.

eight lines which meet b; and similarly for (L, c). The three sets of eight lines so obtained are all distinct, and they exhaust the lines on F_3.

Thus in each case, projecting on to the plane of C_4, we get three Steiner complexes which have four syzygetic lines common, and of the other twenty-four bitangents each occurs once in one of the three complexes.

Hence the choice of two pairs of a Steiner complex determines a division of the remaining sixteen lines into two groups of eight lines each, such that each eight make up a Steiner complex with the original four. Since two pairs can be chosen in fifteen ways, there are fifteen ways of dividing the sixteen lines into two groups of this kind.

Since each choice of two pairs determines two more Steiner complexes, we get $15 \times 2 = 30$ Steiner complexes which have four lines common with a given complex; and these four lines are syzygetic.

11.

Consider a division of a double six

$$a_1 a_2 a_3 a_4 a_5 a_6,$$

$$b_1 b_2 b_3 b_4 b_5 b_6,$$

into two groups of six lines which lie on a hyperboloid, *e.g.*,

$$a_1 a_2 a_3 b_4 b_5 b_6 \quad \text{and} \quad b_1 b_2 b_3 a_4 a_5 a_6.$$

The six lines $c_{23} c_{13} c_{12} c_{56} c_{46} c_{45}$ (where c_{12} is the third line in the planes of $a_1 b_2$ and $a_2 b_1$, &c.) make up a double six with either of these groups of six, viz.,

$$\left.\begin{matrix} a_1\ a_2\ a_3\ c_{56} c_{46} c_{45} \\ c_{23} c_{13} c_{12} b_4\ b_5\ b_6 \end{matrix}\right\} \quad \text{and} \quad \left\{\begin{matrix} c_{23} c_{13} c_{12} a_4\ a_5\ a_6. \\ b_1\ b_2\ b_3\ c_{56} c_{46} c_{45}. \end{matrix}\right.$$

Six lines on a hyperboloid can be chosen from a double six in twenty ways, and each such set of six determines another double six. Now consider a group such as $a_1 a_2 a_3 a_4 a_5 b_6$, five elements being chosen from one row and one from the other. They are five skew A lines and their transversal (b_6), and so form half of a two-ten, which is completed by L, and the other five lines $c_{16} c_{26} c_{36} c_{46} c_{56}$ which meet b_6; and these five also meet a_6, so that with L they also complete the two-ten to which $b_1 b_2 b_3 b_4 b_5 a_6$ belong. Six lines of this type can be chosen in twelve ways, so that there are twelve two-tens which have six lines common with a given double six. Projecting on to the plane of the C_4, we get $20 + 12 = 32$ Steiner complexes which have six lines common with a given one; and these six lines are azygetic.

If we start from a Steiner complex, which is the projection of a two-ten, we naturally obtain the same result. The two-ten consists of L, any line c, and the five plane pairs which meet c. If we choose one line from each pair, we get five skew lines; if they have a single transversal c, the other five have two transversals, and *vice versa*. Hence there are the same number of sets of five A and five B lines among the ten which meet c. The whole number of ways of choosing one from each of the five pairs is $2^5 = 32$, and therefore there are sixteen ways of choosing either five A or five B lines. Either L or c can be added to the five as the sixth line, so that there are four possible cases :—

(*a*) L and five A lines. These belong to a unique two-ten, which contains also the transversal of the five lines, and the other five lines which meet it.

(*b*) c and five B lines. These also belong to a unique two-ten, which contains the other five lines which meet c and L.

In both these cases the two-ten is the original one under discussion.

(*c*) L and five B lines. One of the two transversals determines the original two-ten, the other determines a two-ten which has these six lines common with it.

(*d*) c and five A lines. These belong also to a double six, as in the first case considered.

Hence, by projection, (*c*) and (*d*) each give sixteen Steiner complexes which have six lines common with a given one.

We have now accounted for $1+30+32 = 63$ Steiner complexes, and there are no more. Hence, any two of them have common either four lines forming a syzygetic group or six lines forming an azygetic group.

Six lines, one from each pair of a Steiner complex, can be chosen in $2^6 = 64$ ways. Thus in half the number of cases the six lines belong to a second Steiner complex, and in the other half only to the first; for in any cases which are not explicitly considered above, it is easily seen that six lines belonging to either a double six or a two-ten determine it uniquely.

12.

Six bitangents which touch a conic form a *Brianchon set*. The six lines which are the projection of six lines on a hyperboloid will touch a conic, so that we get, at any rate, three hundred and sixty such sets among any twenty-seven of the bitangents. We can determine the whole number as follows.

Consider a Steiner complex, which is the projection of a double six,

$$a_1 a_2 a_3 a_4 a_5 a_6,$$
$$b_1 b_2 b_3 b_4 b_5 b_6.$$

Choose one of the lines, say a_1, as L, and construct the appropriate cubic surface. Now $a_1 a_2 a_3 b_4 b_5 b_6$ belong also to a second Steiner complex, and, since a_1 is L, the other five must project into five B lines; consequently b_1 must be one of the transversals of these five lines (the second transversal will give the other Steiner complex to which $L a_2 a_3 b_4 b_5 b_6$ belong), and $b_2 b_3 a_4 a_5 a_6$ are the other five lines which meet b_1. Hence six lines which touch a conic will project into either L and five B lines or five A lines and their transversal; and, conversely, these sets of lines must project into lines which touch a conic. Thus any six bitangents which belong to two Steiner complexes touch a conic.

There are on F_3 three hundred and sixty groups of six lines which lie on a hyperboloid, four hundred and thirty-two sets of five A lines, and two hundred and sixteen sets of five B lines. Each of these gives a Brianchon set, so that there are altogether $360+432+216 = 1008$ Brianchon sets.

13.

We deduce as a property of F_3, that if we take five skew lines on it and construct the quadric cone whose vertex is any point P on the surface and which touches the planes joining P to each of the five lines, then if the five lines have a single transversal, the plane joining it to P will touch the cone, and if they have two transversals, the tangent plane at P to F_3 will also touch the cone.

14.

We can express the result for the bitangents in another form. Denote the two-ten into which

$$a_1 a_2 a_3 b_4 b_5 b_6,$$
$$b_1 b_2 b_3 a_4 a_5 a_6,$$

was projected by

$$L\, a_2 a_3 a_4 a_5 a_6,$$
$$\beta \beta_2 \beta_3 \beta_4 \beta_5 \beta_6,$$

so that L corresponds to a_1, a_2 to a_2, a_4 to b_4, ... Then $a_2 a_3 a_4 a_5 a_6$ are five B lines, $\beta_2 \beta_3 \beta_4 \beta_5 \beta_6$ are five A lines, and all the ten meet β. Amongst the bitangents, the six β's, or L and the five a's, touch a conic. Since the two pairs $(a_1 b_1)$, $(a_2 b_2)$, or $(L\beta)$, $(a_2 \beta_2)$, are syzygetic, a_2 and β_2 must intersect. Similarly $(a_3 \beta_3)$, $(a_4 \beta_4)$, $(a_5 \beta_5)$, $(a_6 \beta_6)$ intersect. In order

to obtain another set of five B lines, we have to replace any two or four α's by the corresponding β's; so that if we choose six lines, one from each pair of the two-ten, so that an even number are from each row, we shall have either L and five B lines, or β and five A lines, according to whether L or β is chosen from the first pair.

Hence, if the lines of a Steiner complex are arranged in two rows, so that the six lines in each row touch a conic and the two lines of any pair are in the same vertical column, then six lines, chosen one from each pair, and an even number from each row, will touch a conic.

15.

We have seen that three azygetic lines lines are the projection of

(a) Three skew lines.

(b) Two skew lines and one line which meets both.

(c) Two skew lines and L.

Any azygetic group, obtained by combining these, will project into a set of lines on F_3, such that any three of them have either two or no intersections, *i.e.*, either none of the lines intersect, or, if two do, then all the others meet one of the pair; and no three lie in a plane. [If the group includes L, then none of the other lines intersect.] We thus see that four azygetic lines are the projection of:—

(a) Four skew lines.

(b and c) Three skew lines and a common transversal or L.

(d) A skew quadrilateral.

Five azygetic lines are the projection of:—

(a and b) Four skew lines and a common transversal or L.

(c) Three skew lines and two lines which meet all three.

(d and e) Five skew lines, either A or B.

Six azygetic lines fall into two classes, viz.:—

(α) Those which touch a conic.

(β) Those which do not.

For (α) we have:—

(a) Six lines on a hyperboloid.

(b) Five skew A lines and their transversal.

(c) Five skew B lines and L.

For (β) we have :—

(a) Six skew lines.

(b) Five skew A lines and L.

(c) Five skew B lines and one transversal.

(d) Four skew lines and their two common transversals.

Any other set of six lines will give at least one syzygetic triplet.

For seven azygetic lines we get only two cases :—

(a) Two skew lines and the five B lines which meet them.

(b) Six skew lines and L ;

and it is impossible to find more than seven lines whose projections will be azygetic.

16.

Five azygetic lines determine a Brianchon set uniquely. In the cases enumerated above, the sixth line is :—

(a) The fifth line which meets the transversal, but none of the other four, and makes with them five A lines.

(b) The fifth line which meets both common transversals of the four.

(c) The third line which meets the three skew lines.

(d) The common transversal of the five A lines.

(e) L ;

and it is clear that in each case the line determined is unique.

17.

Seven bitangents such that no three have their points of contact on a conic form an *Aronhold seven.** No six of the seven lines touch a conic. There are on F_3 seventy-two sets of six skew lines, and two hundred and sixteen pairs of skew lines ; each pair with the five lines which meet both projects into an Aronhold seven, so that there are $216+72 = 288$ Aronhold sevens.

* Aronhold, *Berliner Monatsberichte*, 1864.

18.

If we take any five lines of an Aronhold seven, a conic will touch them and one other bitangent, which we have seen is definite and unique. When the Aronhold seven is the projection of six skew lines and L, we have two possible types of a set of five lines, viz., four skew lines and L (these can be chosen in fifteen ways), or five skew A lines (in six ways), making up the twenty-one possible sets of five. We thus determine twenty-one lines, which are the fifteen lines each of which meets two only of the given six lines, and the six lines each of which meets five of the given six, and which with them make up a double six. These twenty-one lines are all distinct, and they exhaust the lines on F_3.

When the Aronhold seven is the projection of two skew lines and their five transversals, five lines chosen from it are of three types :—

(*a*) Five skew B lines (which can be chosen in only one way).

(*b*) Four skew lines and one transversal (in ten ways).

(*c*) Three skew lines and two transversals (in ten ways).

The sixth line is in each case respectively :—

(*a*) L.

(*b*) The fifth line which meets the given transversal, but not the other, nor any of the first four lines.

(*c*) The third line which meets the three skew lines.

We thus obtain, besides L, twenty lines which are all distinct, and with the original seven make up all the lines on F_3.

Thus, when we have chosen an Aronhold seven, each of the other lines is associated with five of the seven and forms an azygetic triplet with the remaining two.

19.

This arrangement of the bitangents gives the simplest notation for them. If we denote the lines of an Aronhold seven by the numerals 1 to 7, then (67) will denote the bitangent which touches the same conic as 1, 2, 3, 4, 5, and so on, the twenty-one combinations of the seven numerals two and two each denoting a single line. If we introduce another numeral 8 and call the lines of the Aronhold seven 18, 28, 38, ..., we get a perfectly symmetrical notation, which corresponds exactly to the Hesse-Cayley algorithm,* by which the twenty-eight lines are symbolically represented by the lines joining eight points.

* Hesse, *Crelle's Journal*, Bd. xlix, p. 264 ; Cayley, *ibid.*, Bd. lxviii and lxxxvii.

20.

A group of six azygetic bitangents which do not touch a conic belongs to a single Steiner complex and also determines an Aronhold seven. If we consider simultaneously the two groups of six which make up a Steiner complex, we get the following cases:—

(a) For a double six there are two types, (α) two groups of six skew lines, either of which with L makes an Aronhold seven, and (β) two groups, each consisting of four skew lines and their two transversals, such as $a_1a_2a_3a_4b_5b_6$ and $b_1b_2b_3b_4a_5a_6$. The fifth line which meets the two transversals b_5b_6 meets also a_5a_6, since it is the line of intersection of the planes (a_5b_6) and (b_5a_6); and therefore the same line completes an Aronhold seven with either group of six lines.

(b) For a two-ten one of the groups consists of L and five skew A lines, the other of their transversal and the other five (B) lines which meet it. The second transversal of the five B lines meets none of the five A lines, and so it completes an Aronhold seven with either group.

Thus in every case, if we divide a Steiner complex into two azygetic groups of six lines which do not touch a conic, the same line must be added to each group to make it into an Aronhold seven. We have seen that such a group of six lines can be chosen in thirty-two ways, *i.e.*, there are sixteen pairs of these groups, and so we get sixteen bitangents associated with the Steiner complex. Since the sixteen are evidently all distinct, we have in this way accounted for all the bitangents.

21.

Now take any six lines of an Aronhold seven. The other six which belong to the same Steiner complex form an Aronhold seven with the same seventh line. Hence an Aronhold seven has seven others associated with it, each of which has one line common with it. Consider two of these. They belong to two Steiner complexes which have five lines of the Aronhold seven common, and therefore they must have a sixth line common. This line belongs to both the associated Aronhold sevens in question, so that any two of these have a single line in common. If we take any one of them, each of the others will have a different line common with it, so that the relation of any seven of them to the eighth is the same, and the eight form a perfectly symmetrical group. Hence the Aronhold sevens fall into groups of eight ("octuples"), and there are $288 \div 8 = 36$ of these groups. There are fifty-six lines in an octuple, and

any bitangent occurs in only two of the Aronhold sevens in it, so that every bitangent must be used twice in an octuple. Hence each bitangent occurs in seventy-two Aronhold sevens.

This result is also obvious from the fact that when a given bitangent is chosen as L, seventy-two Aronhold sevens (corresponding to the seventy-two sets of six skew lines on F_3) which contain L are obtained, and none of the other two hundred and sixteen contain L.

22.

Five azygetic lines determine an Aronhold seven uniquely. For instance, if the five lines are projected into five skew B lines, then the other two lines are the two transversals. If, then, we take five lines of a Brianchon set, we get two more lines which with the five make up an Aronhold seven, so that a Brianchon set has six pairs of lines associated with it.

Project a Brianchon set into six lines on a hyperboloid, $a_1 a_2 a_3 b_4 b_5 b_6$, where each a meets each b. The pair of lines corresponding to any five of these, say $a_1 a_2 a_3 b_4 b_5$, are the other two which meet the two generators $b_4 b_5$ of the same system, but not b_6. Of these two, one belongs to each of the sets of six which makes a double six with the first six lines (with the same notation as before, they are a_6 and c_{45}); and the six pairs of lines so determined are all distinct. Hence these twelve lines themselves form a double six. We have now eighteen lines, forming three groups of six lines each, such that each group lies on a hyperboloid, and any two of the three groups form a double six; these eighteen lines form a *triple six*.* The eighteen lines fall into triplets; each triplet consists of one line from the first hyperboloid, and the two, each belonging to one of the other sets of six, which are obtained by omitting the first (*e.g.*, b_6, a_6, c_{45}); and none of these three intersect, so that they project into an azygetic triplet.

If we denote the eighteen lines by

$$a_1\ a_2\ a_3\ a_4\ a_5\ a_6,$$
$$\beta_1 \beta_2 \beta_3 \beta_4 \beta_5 \beta_6,$$
$$\gamma_1\ \gamma_2\ \gamma_3 \gamma_4\ \gamma_5 \gamma_6,$$

in which the six lines in one row lie on a hyperboloid, and any two rows form a double six, then the three elements in any vertical row form a skew triplet, and any set of seven lines of the type $a_1 a_2 a_3 a_4 a_5 \beta_6 \gamma_6$ projects into an Aronhold seven. Thus a triple six contains eighteen Aronhold sevens.

* German, *Dreisechs*; Italian, *Terna*.

Thus five azygetic bitangents determine the following groups, and each without ambiguity :—

(*a*) An Aronhold seven.

(*b*) A Brianchon set.

(*c*) The Steiner complex, which with (*b*) makes up :—

(*d*) A triple six.

(*e*) A Kummer set, *i.e.*, the sixteen lines left when the Steiner complex (*c*) is taken away.

(*f*) A Caporali set, *i.e.*, the ten lines left when the Brianchon set (*b*) is taken away from the Kummer set (*e*).

23.

A division of a double six into two groups of six lines which lie on a hyperboloid determines a triple six. This can be done in ten ways, and each triple six is found when we begin with any of the three double sixes which it contains. Thus there are $36 \times 10 \div 3 = 120$ triple sixes on F_3. For the quartic, a Steiner complex can be divided into two groups of six lines which touch a conic in sixteen ways, so that there are

$$63 \times 16 \div 3 = 336$$

triple sixes among the bitangents.

24.

A triple six contains three Steiner complexes, and so a group of six azygetic bitangents which do not touch a conic can be chosen from it in $3 \times 32 = 96$ ways. If we project one of these groups of six into six skew lines, the six lines which with them complete the Steiner complex will become six more skew lines, forming a double six with the first six ; and the remaining six lines of the triple six must lie on a hyperboloid.

If the double six containing the first six lines is

$$a_1 a_2 a_3 a_4 a_5 a_6,$$

$$b_1 b_2 b_3 b_4 b_5 b_6,$$

and the other six lines are

$$c_{23} c_{13} c_{12} c_{56} c_{46} c_{45},$$

then each of the groups

$$b_1\,b_2\,b_3\,c_{56}\,c_{46}\,c_{45} \quad \text{and} \quad c_{23}\,c_{13}\,c_{12}\,b_4\,b_5\,b_6$$

consists of six skew lines; *i.e.*, the triple six on F_3 falls into three groups of six skew lines, such that if we choose one group, the double six to which it belongs is completed by a skew triplet from each of the other two groups. These two triplets taken together are six skew lines; and the other twelve in the same way fall into two groups of six skew lines each. We now have all the possible sets of six skew lines which can be made with the triplets $(a_1a_2a_3)$, $(a_4a_5a_6)$, $(b_1b_2b_3)$, $(b_4b_5b_6)$, $(c_{23}c_{13}c_{12})$, $(c_{56}c_{46}c_{45})$.

If we denote these six triplets by $\alpha\beta'\beta\alpha'\gamma\gamma'$, where each pair of triplets (such as $\alpha\alpha'$) lies on a hyperboloid, and any two pairs form a double six, then each line of α meets each line of α', none of β' or γ', and two of β or γ; and similarly for any other triplet. Hence the three triplets $(\alpha\beta\gamma)$, $(\alpha'\beta'\gamma')$, lie on a pair of conjugate trihedra.*

25.

Nine lines which lie on a pair of conjugate trihedra, or their projections, form a Ciani's triple three (Timerding,† "Cianische Dreidrei"). If we denote the nine lines of a triple three on F_3 by

$$\left.\begin{matrix} a_1\,b_2\,c_3 \\ b_3\,c_1\,a_2 \\ c_2\,a_3\,b_1 \end{matrix}\right\} \quad \text{or} \quad \left\{\begin{matrix} a_1\,b_1\,c_1, \\ a_2\,b_2\,c_2, \\ a_3\,b_3\,c_3, \end{matrix}\right.$$

where two lines denoted by the same suffix or the same letter do not intersect, and two denoted by different letters and suffixes do, then in the first scheme three lines of the same row or column do intersect and in the second they do not. *I.e.*, the nine lines of a triple three can be arranged in three skew triplets or three plane triplets, and in each case in two ways.

The double sixes determined by

$$\left.\begin{matrix} a_1\,b_1\,c_1 \\ a_2\,b_2\,c_2 \end{matrix}\right\} \quad \text{and} \quad \left\{\begin{matrix} a_1\,b_1\,c_1, \\ a_3\,b_3\,c_3, \end{matrix}\right.$$

which we may call (12) and (13), have three skew lines $a_1\,b_1\,c_1$ common, so

* See Sturm, *Op. cit.*, Kap. I, and below (§ 25).

† *Crelle's Journal*, Bd. CXXII.

that the other three lines which lie on the same hyperboloid as $a_1 b_1 c_1$ must also be common to both. We thus get three more skew triplets, each of which is common to two of the double sixes (12), (13), (23), and such that any two of them are two pairs of one of these double sixes. Hence the nine lines intersect in exactly the same way as the first nine; and therefore they also lie on a pair of conjugate trihedra and are a Ciani's triple three. The eighteen lines belong to three double sixes [viz., (12), (13), (23)], any two of which have six lines common, and so are a triple six.

If we divide the triple six into skew triplets by taking columns (as $a_1 a_2 a_3$) instead of rows (as $a_1 b_1 c_1$), we shall get, in the same way, three double sixes and a third triple three. The two double sixes determined by

$$\left.\begin{matrix} a_1 a_2 a_3 \\ b_1 b_2 b_3 \end{matrix}\right\} \quad \text{and} \quad \left\{\begin{matrix} a_1 b_1 c_1, \\ a_2 b_2 c_2, \end{matrix}\right.$$

have four lines $a_1 b_1 a_2 b_2$ common, and therefore they have no other common line; and similarly for any other two double sixes, one of each type. Thus the lines of the third triple three must be all different from those of the second, and so each of the twenty-seven lines on F_3 belongs to one only of the three triple threes. Call the first triple three α, the other two β and γ. If we start with β, we shall find α for one of the triple threes associated with it, and so the other must be γ; *i.e.*, the relation between any two of the three α, β, γ is the same.

Hence the twenty-seven lines on F_3 fall into three Ciani's triple threes, and any two of these form a triple six. A skew triplet can be chosen in seven hundred and twenty ways, and each determines a triple three; but each triple three contains six skew triplets, so that there are $720 \div 6 = 120$ triple threes (the number of pairs of conjugate trihedra), and the twenty-seven lines fall into three triple threes in $120 \div 3 = 40$ ways. Hence any given line occurs in forty triple threes.

The number of triple threes among the bitangents will be determined below (§ 27).

26.

The ten bitangents made up of a triple three and L are the ten lines which are left when a triple six is taken away, and they are called a *Caporali set*. A triple six which does not contain L can only be the projection of a triple six on F_3, while one which does contain L is the projection of five skew B lines, their two transversals, and the ten lines of which five meet each of the two transversals. Hence, in the first case, a

Caporali set is the projection of a triple three on F_3 with L, and in the second, of the ten lines which meet neither of two skew lines. If, therefore, we take each line of a Caporali set in turn as L, the other nine will be a triple three. A triple three on F_3 consists of nine lines lying in three planes, forming three triangles in perspective, and they can be arranged thus in two ways. Perspective properties are unaltered by projection, and therefore a Caporali set may be described as consisting of ten lines, any nine of which form three triangles in perspective in two different ways.

27.

Any triple three has thus associated with it two other triple threes, and a single line which with any of the three triple threes makes up a Caporali set. Corresponding to any given L, we get one hundred and twenty triple threes, and those found by taking two different lines as L are all different. Hence there are $120 \times 28 = 3360$ triple threes. Each of them determines a Caporali set; but since any Caporali set contains ten triple threes (by omitting each line in turn), there are $3360 \div 10 = 336$ Caporali sets.

On F_3, we get a Caporali set for each pair of skew lines, so that there are two hundred and sixteen such sets. Now take each bitangent in turn as L. A given Caporali set will be obtained when any of the sixteen bitangents which do not belong to it is taken as L, so that we get $216 \times 28 \div 16 = 336$ Caporali sets for the C_4.

If we consider only a single cubic surface, we get 120 from conjugate trihedra + 216 from pairs of skew lines = 336 Caporali sets.

A triple six can be divided into two triple threes in ten different ways. For any nine of the remaining ten bitangents form a triple three, and so determine a division of the triple six into two triple threes. Hence, as before, since there are 336 triple sixes, there are $336 \times 10 = 3360$ triple threes, and the number of Caporali sets is the same as the number of triple sixes, viz., 336.

28.

Corresponding vertices of three plane triangles of a triple three on F_3 lie on three straight lines (the edges of one of the trihedra on which the nine lines lie), and these three lines meet in a point (the vertex of the trihedron). Projecting, and taking each trihedron in turn, the points of intersection of the bitangents in a triple three lie by threes on six straight

lines (Aronhold lines*) which meet by threes in two points (Geiser points).

If we consider one set of three triangles in perspective which make up a triple three, three of the Aronhold lines are the axes of perspective of the triangles taken two and two, and corresponding vertices of the triangles lie on the other three; each set of three Aronhold lines meets in a Geiser point, which in the second case is the centre of perspective of the triangles.

Six lines on F_3 which lie in two planes project into a "de Paolis double three,"† and the line of intersection of their planes projects into an Aronhold line. The three lines joining corresponding vertices of the two triangles are Aronhold lines for any one of the triple threes composed of the double three and three out of the four lines which belong to the same Caporali set. Thus any Aronhold line belongs to four triple threes, *i.e.*, four Geiser points lie on it; and any given double three will be obtained when any of the four lines which complete its Caporali set is taken as L. Therefore, since there are 3360 triple threes, and each contains six double threes, there are $3360 \times 6 \div 4 = 5040$ double threes, and the same number of Aronhold lines. Three Aronhold lines go through each Geiser point. Each triple three has two Geiser points, corresponding to the vertices of the two trihedra of a triple three on F_3; hence there are $3360 \times 2 = 6720$ Geiser points.

29.

The four lines which with a double three make up a Caporali set are syzygetic, since in the case considered above they correspond to three plane lines and L; and if we take away four syzygetic lines from a Caporali set, the six which are left are a double three. This double three is made up of three pairs, each of which forms a syzygetic triplet with any one of the other four lines in the Caporali set, and the intersections of these three pairs are collinear and lie on the Aronhold line of the double three. Now consider the other cases which may occur. First, when the Caporali set does not contain L, on F_3 it consists of the ten lines which meet neither of two skew lines $a_1 a_2$. Of these ten lines, any one meets three of the others and not the remaining six, and no three lie in a plane. If we choose two plane lines, there are four others which meet neither of them, and these four themselves form two plane pairs, such that

* Aronhold, *loc. cit.* (*Berliner Monatsber.*, 1864), p. 505. See also references in § 34.

† Timerding, *loc. cit.*

a line of one pair meets neither of the other pair; *i.e.*, the three pairs all meet the same line; and there are fifteen ways of choosing four lines which project into four syzygetic bitangents, *i.e.*, two pairs of this kind. The common transversal β of the two pairs is one of the five lines which meet both α_1 and α_2, and so does not belong to the Caporali set. The remaining six lines of the set consist of :—

(*a*) The third plane pair $\gamma_1\gamma_2$ (*i.e.*, not containing either α_1 or α_2) which meets β.

(*b*) Four non-intersecting lines, each of which meets either γ_1 or γ_2, and two of the first four lines, one from each pair. The pair $\gamma_1\gamma_2$, or the two lines, both of which meet either γ_1 or γ_2, project into lines which are syzygetic with any one of the first four. Thus they are the three pairs into which the double three must be divided in order to find its Aronhold line.

In the first case considered (*i.e.*, when the double three was the projection of two plane triangles), the lines joining the point of projection P to the intersections of the three pairs of bitangents met the plane of C_4 in collinear points; *i.e.*, they were coplanar. Corresponding to this, the plane containing the transversals from P to two pairs of skew lines, none of which intersect, must pass through the point of intersection of two lines on F_3, each of which meets one of the pairs of skew lines. (It can easily be verified that if we divide four skew lines a, b, c, d on F_3 into two pairs ab, cd, there are two, and only two, lines on F_3, which intersect, and of which one meets ab and the other cd.)

In order to make a cubic surface contain six lines of this kind we have to satisfy nineteen conditions, viz., four for each of the skew lines, one for the point of intersection of the transversals, and one more for each transversal; and thus the surface is unique. Accordingly the above result may be stated as follows :—The locus of a point which moves so that the plane containing the transversals from it to two pairs of lines, none of which intersect, passes through a fixed point, is the cubic surface which contains the four given lines and the transversals drawn from the fixed point to each of the pairs.

When the Caporali set contains L, we have considered the case which arises when the four syzygetic lines are L and three plane lines, so that the double three merely consists of six lines forming two plane triangles. We can also take as the four lines two plane pairs which meet a fifth line. This fifth will belong to the same triple three, and none of the four remaining lines of the triple three meet it, while they form two skew pairs, such that each line of one pair meets each of the other pair. Thus the

double three in this case consists of a skew quadrilateral, a line which meets none of its four sides (this line is unique on F_3), and L; the three pairs are the two pairs of opposite sides of the quadrilateral, and L and the remaining line. We may call this line the *antigram* (Sturm,* "Gegengerade") of the skew quadrilateral. Thus each skew quadrilateral has a single antigram, and each line is the antigram of forty skew quadrilaterals, since it occurs in forty triple threes, each of which contains one of the quadrilaterals. Hence there are $27 \times 40 = 1080$ skew quadrilaterals on F_3.

With the notation used in § 25, the four syzygetic lines which we have chosen are $b_1 c_1 b_2 c_2$. These all meet a_3, and of the other four $a_1 a_2 b_3 c_3$, none of which meet a_3, a_1 and a_2 both meet b_3 and c_3, so that the four lines form a skew quadrilateral of which a_3 is the antigram.

The existence of an Aronhold line for the double three leads to the following result:—The plane containing the two transversals from any point P on F_3 to the two pairs of opposite sides of a skew quadrilateral lying on the surface passes through the point in which the antigram of the quadrilateral meets the tangent plane at P to F_3.

30.

We have now considered all the possible sets of lines on F_3 which can project into double threes. To find the number of sets of each type, we have first, two plane triangles which have no side common, of which there are 720 pairs; second, four skew lines, which can be chosen in 1080 ways, and each four can be divided into pairs in three ways, each division having a corresponding pair of intersecting transversals; thirdly, skew quadrilaterals, of which there are 1080. Hence we have, as before,

$$720 + 3 \times 1080 + 1080 = 5040 \text{ double threes.}$$

31.

The sixteen bitangents which are left when the twelve lines of a Steiner complex are taken away form a *Kummer set.* The number of Kummer sets is the same as that of Steiner complexes, viz., 63.

We have seen that if we choose four syzygetic bitangents, the remaining twenty-four fall into three groups of eight, any one of which makes up a Steiner complex with the first four lines; *i.e.*, any two of the three

* *Op. cit.*, p. 62.

groups of eight lines make up a Kummer set. If then we take a Kummer set, the choice of two pairs from the corresponding Steiner complex determines two more Kummer sets which have eight lines common with the first. The two pairs of lines can be chosen in fifteen ways, so that thirty Kummer sets have eight lines common with a given one, and these eight lines are four pairs of a Steiner complex.

Consider a Steiner complex, and six lines which with it make up a triple six. The ten lines which are left form a Caporali set, and they make a Kummer set with any one of the three Brianchon sets in the triple six. We thus get three Kummer sets which have ten lines common to all three. A Steiner complex determining a Kummer set, belongs to sixteen triple sixes, and each determines two Kummer sets besides the first. Hence there are thirty-two Kummer sets which have ten lines common with a given one, these ten lines form a Caporali set, and there are sixteen ways of dividing a Kummer set into a Brianchon set and a Caporali set.

32.

There are two cases to consider on F_3. First, corresponding to a double six we get L and fifteen lines which form Cremona's "Polar Hexahedron."* The properties of these lines have been fully discussed by Cremona,† who shewed that their projections on any plane have the same properties (of collinear intersections, &c.) as the fifteen lines which join six points on a conic, and by H. W. Richmond,‡ who derives both sets of lines (*i.e.*, in two and three dimensions) from a figure in space of four dimensions. We may call the fifteen bitangents a *generalised hexagram*. If any one of the lines of a Kummer set of bitangents is chosen as L, the Steiner complex consisting of the other twelve lines projects into a double six, and therefore the remaining fifteen lines of the Kummer set project into a Cremona hexahedron. That is to say, any fifteen lines of a Kummer set form a generalised hexagram.§

Since two Kummer sets have only either eight or ten lines common, the fifteen lines of a generalised hexagram belong to a single Kummer set. Hence $63 \times 16 = 1008$ generalised hexagrams can be chosen from the bitangents.

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* *Math. Annalen*, Bd. XIII.

† *Atti della R. Accad. dei Linc.*, Ser. III, t. I, 2 (1877), p. 854.

‡ *Quarterly Journal*, Vols. XXXI and XXXIV.

§ H. W. Richmond (*Quarterly Journal*, Vol. XXXI, p. 156) gives this result as a consequence of the symmetry of the Hesse-Cayley algorithm of the Kummer set.

33.

Secondly, if we choose one line g on F_3, then g, the ten lines which meet it, and L, project into a Steiner complex, and the sixteen lines which do not meet g project into a Kummer set. It will be convenient to speak of these sixteen lines on F_3 also as a Kummer set.

If we choose three lines forming a plane triangle, eight of the remaining twenty-four meet each of them, and the sixteen lines which meet any two of the three form a Kummer set. If we call one of the lines a, the corresponding Kummer set consists of the eight pairs of lines which meet any pair of lines coplanar with a. This gives five ways of dividing the set into a pair of groups of eight lines each, such that each group of a pair is made into a two-ten by the addition of the same four lines, viz., the three coplanar lines by which the pair was determined and L. Now consider two pairs of lines which meet a, say b_1c_1, b_2c_2. These form two pairs of a double six in two ways, viz., (b_1b_2), (c_2c_1), and (b_1c_2), (b_2c_1). The other sixteen lines belonging to the same two double sixes are all distinct, and none of them meet a. Hence we have obtained another division of the sixteen lines into two groups of eight. This can be done in ten ways (viz., the number of ways of choosing two pairs from the five pairs of lines which meet a), so that altogether we get as before the fifteen ways of dividing a Kummer set into two groups of eight lines each, which are made into Steiner complexes by the addition of the same four syzygetic bitangents.

The two Kummer sets defined by two plane lines a, b have eight lines common, viz., the eight which meet the third line in the plane of ab. The two sets defined by two skew lines c, d have ten lines common, viz., the ten lines which meet neither c nor d. The twelve lines each of which belongs to one only of the two Kummer sets are the two lines c, d, and the ten lines each of which meets one only of these two; these twelve lines form a double six. The two Kummer sets together account for twenty-two lines, the five which are left being the five which meet both c and d. These five lines, and the ten lines which meet neither c nor d, and are common to the first two Kummer sets, form a Cremona hexahedron, and with L project into a third Kummer set of bitangents.

34.

The most important references to Geiser's method are as follows. (Several of them have already been referred to above.)

1. Geiser, besides the original memoir,* states† the two kinds of sets of lines on F_3 (see § 7) which project into a Steiner complex. He makes the proof of this depend on the theorem that the lines which cut three pairs of lines in point-pairs of a system in involution, envelope a curve of the third class.

2. Zeuthen‡ uses the properties of plane quartics to demonstrate, by Geiser's method of projection, Schläfli's theorems on the number of *real* straight lines and triple tangent planes of cubic surfaces, and their resulting division into species.

3. Cremona§ deduces the properties of Pascal's hexagram from those of the fifteen lines which lie on a cubic surface with a conical point, and which do not pass through that point. By projecting these lines from the conical point they become the fifteen lines which join six points on a conic, and by projecting from any other point they become the fifteen lines which I have called a "generalised hexagram" (§ 32). The properties discussed depend only on the fact that the lines lie by threes in fifteen planes, and so Cremona extends his results (of collinearity of intersections, &c.) to the projection of a Cremona hexahedron on any cubic surface.

4. Salmon‖ refers to Geiser, but does not mention any details.

5. Ciani¶ and Timerding** both make use of Geiser's method in order to explain and find the number of Aronhold lines.

6. H. W. Richmond's†† use of it has already been mentioned (§ 32).

7. G. Kohn‡‡ mentions Geiser's method in a summary of the various ways of generating a plane quartic, and makes use§§ of the Hesse-Cayley algorithm, and the fact that six skew lines on a cubic surface, and L, project into an Aronhold seven, in order to establish a correlation be-

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* *Math. Annalen*, Bd. I (1869).
† *Crelle's Journal*, Bd. LXXII (1870), p. 376.
‡ *Math. Annalen*, Bd. VIII (1876), p. 1.
§ *Atti della R. Accad. dei Linc.*, Ser. III, t. I, 2 (1877), p. 854.
‖ *Higher Plane Curves*, § 259, note.
¶ *Rend. Ist. Lomb.*, Vol. XXVIII (1895), p. 683.
** *Crelle's Journal*, Bd. CXXII (1900), p. 225.
†† *Quarterly Journal*, Vol. XXXI (1900), p. 155.
‡‡ *Encykl. der Math. Wiss.* (1909), III, c. 5, § 71.
§§ *Sitzungsber. der Akad. Wien*, Bd. CXIV (1905), p. 1457.

tween the eight associated sevens of an octuple (§ 21) and eight sets of seven points in a plane.

8. Of minor references, the most interesting is by Clebsch,* in an article on the representation of a cubic surface on a plane, in which he points out that the "Uebergangscurve," or double curve, each point of which corresponds to two consecutive points on the cubic surface, is simply Geiser's quartic.

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* *Math. Annalen*, Bd. III.

ON DIFFERENTIAL EQUATIONS WITH FIXED BRANCH POINTS

By M. J. M. HILL *and* ARTHUR BERRY.

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1. The object of this note is to draw attention to a point in Fuchs' paper "Über Differentialgleichungen deren Integrale feste Verzweigungspunkte besitzen" (*Berliner Sitzungsberichte*, 1884, pp. 699–710), and *Gesammelte Mathematische Werke*, Vol. 2, No. XLIII (1906). Fuchs treats the equation

$$F\left(z, y, \frac{dy}{dz}\right) = 0, \tag{A}$$

which is rational and integral with regard to y and dy/dz. It is supposed to be irreducible, *i.e.*, it cannot be resolved into equations of lower degree in dy/dz with coefficients which are rational and integral with regard to y. He then forms the discriminant of

$$F\left(z, y, \frac{dy}{dz}\right) = 0,$$

treated as an equation for dy/dz. This he calls

$$D(z, y) = 0. \tag{C}$$

He supposes that $y = \eta$ is a root of this discriminant equation, and that ζ is the value of dy/dz, which satisfies simultaneously

$$F(z, y, y') = 0, \tag{A}$$

$$\frac{\partial F(z, y, y')}{\partial y'} = 0, \tag{B}$$

where

$$y' = \frac{dy}{dz}.$$

Fuchs arrives at the following conclusions (p. 707):—

"Die nothwendigen und hinreichenden Bedingungen dafür, dass die Integrale der Gleichung

$$F(z, y, y') = 0 \tag{A}$$

feste, sich nicht mit den Änderungen der Anfangswerthe stetig verschiebende Verzweigungspunkte besitzen, sind die folgenden :

" (i) Die Gleichung (A) hat die Form

$$y'^m+\psi_1 y'^{m-1}+\psi_2 y'^{m-2}+\ldots+\psi_m = 0, \qquad (F)$$

worin $\psi_1, \psi_2, \ldots, \psi_m$ ganze rationale Functionen von y mit von z abhängigen Coefficienten von der Beschaffenheit bedeuten, dass ψ_k höchstens vom Grade $2k$ in Bezug auf y ist.

" (ii) Ist $y = \eta$ eine Wurzel der Discriminantengleichung (C), für welche die durch (F) definirte algebraische Function y' von y sich verzweigt, so ist η ein Integral der Gleichung (F). In der y' als algebraische Function von y darstellenden Riemann'schen Fläche hat y' in sämmtlichen über $y = \eta$ liegenden Verzweigungsstellen den Werth

$$y' = \zeta = \frac{d\eta}{dz}.$$

" (iii) Je α Blättern, welche sich in $y = \eta$,

$$y' = \zeta = \frac{d\eta}{dz}$$

verzweigen, entsprechen mindestens $\alpha-1$ mit $y = \eta$ zusammenfallende Wurzeln der Gleichung

$$F(z, y, \zeta) = 0 \qquad (E)$$

mit der Unbekannten y."

2. In attempting to make some examples of equations satisfying these conditions it occurred to one of the writers (M. J. M. Hill) to examine the equation

$$\left[\frac{d\{y-f(z)\}}{dz}\right]^m = \{y-f(z)\}^{m+r},$$

where m and r are positive integers prime to one another and $r < m$. This examination shewed that the equation satisfies all Fuchs' conditions, but that the integral, viz.,

$$y-f(z) = \left\{\frac{r}{m}(c-z)\right\},$$

has a movable branch point,* for which y is infinite. Thus Fuchs' conditions were seen to be insufficient.

* When $r = 1$, there is no branch point, but the condition that r is prime to m is not satisfied.

3. Further examination shews that Fuchs has paid insufficient attention to the possibility that y should be infinite at a branch point. He has, in fact, considered this possibility in establishing his condition (i); for he first shews that if the equation be

$$a_0 y'^m + \ldots + a_m = 0, \tag{1}$$

where $a_0, \ldots, a_m$ are polynomials in y, then a_0 must be independent of y, and can be replaced by unity; he then (p. 706) uses the transformation $w = 1/y$, and, applying the condition that the coefficient of w'^m in the transformed equation should be independent of w, obtains the rest of his condition (i). He does not notice, however, that it is also necessary that the conditions (ii) and (iii) should hold for the transformed equation. Now every factor $y-\eta(z)$ $(\eta \neq 0)$ of the discriminant $D(z, y)$ gives rise to a factor $w-1/\eta(z)$ of the discriminant $D'(z, w)$ of the transformed equation, and for such factors the conditions (ii) and (iii) are automatically satisfied for the transformed equation, if they are satisfied for the original equation; but D' may also contain a factor w,* and the corresponding conditions may, or may not, be satisfied.

Fuchs' conditions have therefore to be supplemented by a fourth condition.

(iv) If the equation be transformed by the substitution $w = 1/y$, then conditions (ii), (iii) must hold at the branch points (if any) of w' regarded as a function of w, corresponding to $w = 0$.

4. Thus in the example given in § 2, the transformed equation is

$$[w' + w^2 f'(z)]^m = (-)^m w^{m-r} [1 - w f(z)]^{m+r};$$

$w = 0$ is a factor of the discriminant; condition (ii) is satisfied, since $w = 0$ is an integral of the transformed equation; but near $w = 0$, we have

$$w' = -w^{(m-r)/m} \ldots,$$

and condition (iii) gives $m - r \geqslant m - 1,$

$$r \leqslant 1,$$

so that (iv) is not satisfied, if $r > 1$.

It is also easy to construct examples in which condition (ii) does not hold for the transformed equation. Take the equation

$$(y' - 3y - y^2)^2 = y(1+3y)^2,$$

* It is easy to see when this case occurs, for the discriminant of (i) is of weight $m(m-1)$, and since a_r is of degree $2r$ at most in y, D is in general of degree $2m(m-1)$ in y; but if D is of degree $2m(m-1)-s$ only, then D' will contain a factor w^s.

which satisfies Fuchs' conditions. The transformed equation is

$$(w'+3w+1)^2 = w(w+3)^2,$$

$w = 0$, $w' = -1$ is a branch-point; but $w = 0$ is not an integral. The integral of the original equation is

$$y = 1 - \frac{2}{\sqrt{(c-z)}} + \frac{1}{c-z},$$

so that there is in effect a movable branch-point.

5. The necessity, which we have pointed out, of employing in problems of this type the transformation $y = 1/w$, or otherwise dealing with the behaviour of the integral when y is infinite, is, however, by no means a new idea. It is very fully recognised by Briot and Bouquet in their much earlier discussion of the equation $f(y', y) = 0$,* and it is quite explicitly stated by Painlevé in his discussion of the problem in his Stockholm lectures (p. 60). On the other hand, we are not aware that any one has pointed out that Fuchs' actual statement is defective, while it has been reproduced in its incorrect form at least in two standard treatises, and one important memoir,† as well as in Fuchs' *Gesammelte Mathematische Werke* (Vol. II, p. 364), so that any one reading the subject for the first time would be extremely likely to be misled.

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* *Journal de l'Ecole Polytechnique*, Vol. 21, cah. 36 (1856), pp. 199–254; *Théorie des Fonctions Elliptiques*, Book V, ch. iv. See, for example, Theorème I, § 241, in the book.

† Forsyth, *Theory of Differential Equations*, Vol. II, § 121; Schlesinger, *Einführung in die Theorie der Differentialgleichungen*, § 67; Poincaré, *Acta Mathematica*, Vol. 7 (1885), pp. 2, 3.

THE GENERATION OF CUBIC CURVES BY APOLAR PENCILS OF LINES

By W. P. Milne.

[Received and Read June 9th, 1910.]

1. In the present paper it is proposed to discuss the following problem :—

"Given a cubic curve, to find two inscribed triangles ABC and DEF, such that if P be any point on the cubic curve, the two pencils of lines $P(A, B, C)$ and $P(D, E, F)$ shall be apolar for all positions of P on the curve."

2. Let $\Sigma \equiv a_l^3 = 0$ and $\Sigma' \equiv a_l'^3 = 0$ be two given class cubics. Then it can easily be shewn that the locus of P moving so that the tangents from P to Σ apolarly separate the tangents from P to Σ' is

$$(aa'x)^3 = 0.$$

Furthermore, it is evident that the apolar locus $(aa'x)^3$ is apolar to both $\Sigma \equiv a_l^3 \equiv \beta_l^3 = 0$ and Σ' since

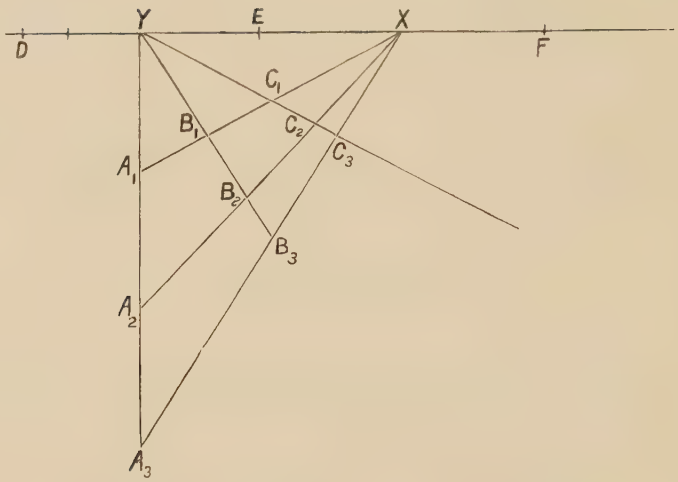
$$(aa'\beta)^3 \equiv 0.$$

Also, if Σ degenerate into the vertices of the triangle ABC, then the fact that the apolar locus of ABC and Σ' is apolar to the degenerate class cubic A, B, C geometrically means that the polar conic with respect to the apolar locus of any one vertex of ABC harmonically separates the other two.

Let us now first of all discuss the case of the apolar locus of two triads, one of which consists of three collinear points D, E, F. Let the Hessian points of D, E, F be X and Y.

Let $A_1B_2C_3$ be the other given triad. Join X and Y to the points A_1, B_2, C_3, and let the lines thus obtained cut as shewn.

Then it is a known property of a collinear triad and its Hessian



quadratic that any point taken on the given line will, along with the Hessian quadratic, apolarly separate the given triad.

Hence $A_2(D, E, F)$ is apolar to $A_2(X, Y, C_3)$,

i.e., $A_2(D, E, F)$,, $A_2(A_1, B_2, C_3)$.

Thus A_2 lies on the apolar locus of DEF and $A_1B_2C_3$. Similarly with A_3, etc.

Hence the apolar locus passes through DEF and the nine points of intersection of XA_1, XB_2, XC_3 along with YA_1, YB_2, YC_3. If, therefore, we take any point O as origin, OX and OY as axes, DEF as the line at infinity, and OD, OE, OF as represented by $x^3+y^3=0$, the equation to the apolar locus will assume the form

$$(x-x_1)(x-x_2)(x-x_3)+(y-y_1)(y-y_2)(y-y_3)=0. \qquad (1)$$

From the symmetry with which x_1, x_2, x_3 and y_1, y_2, y_3 are involved we deduce the following theorem :—

In the above figure the apolar loci of DEF and any one of the six triangles represented by the terms in the expansion of the determinant

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

are the same cubic curve.

Also the apolar locus of ABC and Σ' will pass through A, B and C. For we can find a point P indefinitely near to A such that $P(A, B, C)$ is

apolar to $P\Sigma'$ (where $P\Sigma'$ represents the pencil of tangents from P to Σ), and P, *i.e.*, A (in the limit) must therefore lie on the apolar locus.

We have thus the following proposition :—

" The apolar locus of the two triangles A, B, C and D, E, F passes through the six vertices of the two triangles and is apolar to each triangle."

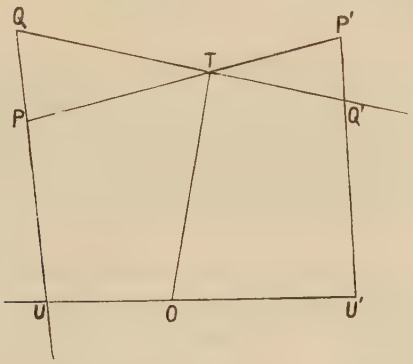
We shall call triangles apolar to a given cubic curve " apolar triangles with respect to the cubic."

3. Let us consider now the following problem :—

" Given a triangle DEF inscribed in a given cubic curve, and apolar to it, to find the whole system of inscribed apolar triangles ABC which along with DEF are capable of apolarly generating the cubic, as described in Art. 1."

4. We require the subjoined theorem :—

If ABC and DEF be two inscribed apolar triangles capable of apolarly generating the cubic, and if we project these two triangles through a point T on the cubic on to the cubic again, thus getting $A'B'C'$ and $D'E'F'$, then will $A'B'C'$ and $D'E'F'$ also be two inscribed apolar triangles capable of apolarly generating the cubic.



Let UOU' be three fixed collinear points on the curve, and let T be a point such that the tangent at T passes through O. Let UPQ be a chord through U cutting the cubic again in P and Q. Let PT and QT meet the curve again in P' and Q'. Then two cubics pass through the nine points U, P, Q, O, T, T, U', P', Q', namely, (1) the given cubic, and (2)

the three lines UOU', PTP', QTQ'. Thus all cubics through eight of the points must pass through the ninth.

Now a third cubic through the eight points U, P, Q, O, T, T, P', Q' consists of the three straight lines QPU, OT and $P'Q'$. Thus this cubic must pass through the ninth point U'. Consequently $P'Q'$ must pass through U'.

Now, if U and U', O and T be regarded as fixed points, a one-to-one algebraic correspondence has been proved to exist between the chords UPQ and $U'P'Q'$.

Thus, if ABC and DEF be two inscribed apolar triangles capable of apolarly generating the cubic, $U(A, B, C)$ will apolarly separate $U(D, E, F)$.

Hence, if A', ... be the projections of A, ... through T, $U'(A', B', C')$ will apolarly separate $U'(D', E', F')$, since UA and $U'A'$, ... have just been proved to be corresponding rays in homographic pencils with U and U' for vertices.

This theorem holds for all points U and U' situated on the cubic, collinear with O.

Hence, if ABC and DEF be capable of apolarly generating the cubic, so will $A'B'C'$ and $D'E'F'$.

Corollary.—Since $A'B'C'$ will along with $D'E'F'$ apolarly generate the cubic, both $A'B'C'$ and $D'E'F'$ must be apolar triangles.

Thus the projection of an apolar triangle through a point on the cubic is an apolar triangle.

5. In order, therefore, to find the complete system of inscribed apolar triangles, co-apolar with a given inscribed apolar triangle (*i.e.*, capable of generating the cubic apolarly along with the given inscribed apolar triangle), it will be sufficient to solve the problem in the case of some one definite projection of the given inscribed apolar triangle through a point on the cubic.

Project the given inscribed apolar triangle through a point on the cubic so that the three projected points shall be collinear.

The question then arises as to whether any given cubic can be reduced to the form (1). This can easily be done if we choose as the line at infinity any tangent to the Cayleyan, as origin any point O, and as axes the lines joining O to the Hessian points of the triad in which this tangent to the Cayleyan cuts the cubic. The equation of the cubic will then assume the form

$$x^3+3px^2+3qx+y^3+3ry^2+3sy+c=0, \qquad (2)$$

which may be written

$$(x^3+3px^2+3qx+\lambda)+(y^3+3ry^2+3sy+c-\lambda)=0, \qquad (3)$$

for various values of λ.

Corresponding to every assigned value of λ we shall get a distinct set of values for x_1, x_2, x_3 and y_1, y_2, y_3.

Hence, corresponding to any given tangent to the Cayleyan cutting the cubic in the triad *DEF*, we get an ∞' number of groups of six triangles co-apolar with *DEF*.

6. Now it is plain from the last article that to assign one of the roots of either

$$x^3+3px^2+3qx+\lambda=0,$$

or

$$y^3+3ry^2+3sy+c-\lambda=0,$$

is tantamount to giving the corresponding value for λ.

Hence, if we be given *one* of the triad of lines through either *X* or *Y*, we can find uniquely the triad of lines through *X*, and also the triad of lines through *Y*.

We have thus the following geometrical construction for finding all triangles co-apolar with the given collinear triad *DEF*, it being known that *DEF* touches the Cayleyan :—

"Take *X* and *Y*, the Hessian points of the given triad *DEF*. Draw through *X* or *Y* any one chord to the cubic, *e.g.*, $YA_1A_2A_3$. Join A_1, A_2, A_3 to *X*, and join the intersections thus found with the cubic to *Y*, getting the figure of Art. 2."

For every assigned position of the chord $YA_1A_2A_3$ we get a group of six triangles co-apolar with *DEF*.

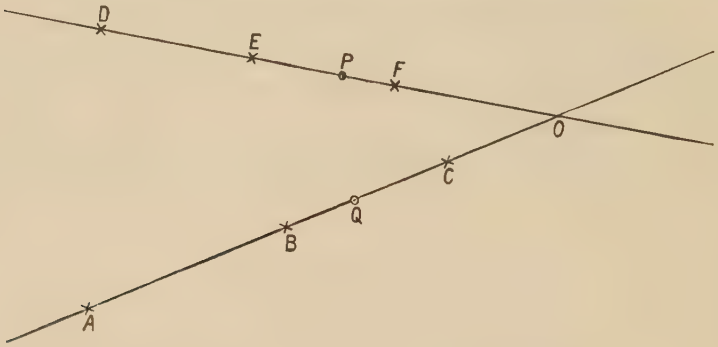
"If we then re-project *DEF* into its original inscribed apolar triangle, and also project any triangle co-apolar with *DEF* through the same point, we shall get the complete system of inscribed apolar triangles co-apolar with the given inscribed apolar triangle."

Corollary.—"If we project an inscribed apolar triangle *ABC* through a point on the cubic into three collinear points *A'*, *B'*, *C'*, then will *A'B'C'* be a tangent to the Cayleyan."

7. "To find the collinear triads co-apolar with a given collinear triad *DEF*, which is the intersection of a tangent to the Cayleyan with the given cubic."

Let *ABC* be a collinear triad co-apolar with *DEF*.

Then the line ABC will also be a tangent to the Cayleyan by Art. 5.



Let the tangent to the cubic at D cut the line ABC in Q.

Let the two lines ABC and DEF intersect in O.

Then $D(E, F, Q)$ is apolar to $D(A, B, C)$, since D is a point on the apolar locus of ABC and DEF.

Hence (O, O, Q) is apolar to (A, B, C). Thus Q is a fixed point.

The tangents at DEF, therefore, all pass through Q. Consequently DEF is part of the polar conic of Q.

Let the tangents at A, B, C all meet in P.

Then since the polar conic of P passes through Q, the polar line of Q will pass through P. But the polar line of a point with respect to a degenerate conic passes through the point of intersection of the lines that compose this degenerate conic. Thus the two lines that make up the polar conic of Q intersect in P and *vice versa*.

Hence P and Q will be corresponding points and ABC and DEF will be portions of the degenerate polar conics of P and Q respectively.

"If P and Q be two 'corresponding points' with respect to a given cubic curve, then either of the lines forming the degenerate polar conic of P will, along with either of the lines forming the degenerate polar conic of Q, cut the given cubic curve in two collinear triads co-apolar with one another."

8. "Inscribed triangles whose vertices are three corresponding points $u+\omega_1$, $u+\omega_2$, $u+\omega_3$."

Consider the harmonic line whose elliptic parameters are ω_1, ω_2, ω_3 where

$$\frac{x}{\wp u} = \frac{y}{\wp' u} = \frac{z}{1}.$$

Then the triangle whose vertices are the points corresponding to the point

u in the three different ways corresponding to the three different elliptic periods $2\omega_1$, $2\omega_2$, $2\omega_3$ may be regarded as the projection through the point $-u$ of the points of intersection of the harmonic line corresponding to the flex whose parameter is zero. But a harmonic line is a tangent to the Cayleyan, and hence the triangle $u+\omega_1$, $u+\omega_2$, $u+\omega_3$ which is a projection of the points of intersection of this line through a point on the cubic, will be an inscribed apolar triangle (Art. 6, Corollary).

We wish to shew that two triangles of this species *cannot* be co-apolar.

Project one of the triangles into a harmonic line. We wish to shew that no triangle of the species $u+\omega_1$, $u+\omega_2$, $u+\omega_3$ can be co-apolar with the points of intersection of the harmonic line ω_1, ω_2, ω_3.

Now the form of the equation to the apolar locus found in Art. 2,

$$(x-x_1)(x-x_2)(x-x_3)+(y-y_1)(y-y_2)(y-y_3)=0,$$

shews that if we choose the origin so as to give

$$\Sigma x_1 = 0,$$

$$\Sigma y_1 = 0,$$

the three asymptotes will all meet in the origin, *i.e.*, the centroid of a triangle co-apolar with the line at infinity.

Also, if we choose the above harmonic line to be the line at infinity, the asymptotes will all meet in the point of inflexion whose parameter is zero.

Consider $$x = \frac{\wp u}{\wp' u} \quad \text{and} \quad z = \frac{1}{\wp' u}. \tag{4}$$

If the flex $u=0$ be the centroid of the triangle $u+\omega_1$, $u+\omega_2$, $u+\omega_3$, then will $\Sigma z = 0$, *i.e.*,

$$\frac{1}{\wp'(u+\omega_1)} + \frac{1}{\wp'(u+\omega_2)} + \frac{1}{\wp'(u+\omega_3)} = 0 \quad [\text{from (4)}],$$

i.e., $$\Sigma (e_2-e_3)(\wp u - e_1)^2 = 0,$$

i.e., $$e_1^2(e_2-e_3)+e_2^2(e_3-e_1)+e_3^2(e_1-e_2)=0,$$

i.e., $$(e_2-e_3)(e_3-e_1)(e_1-e_2)=0,$$

which is impossible, since e_1, e_2, e_3 are by hypothesis all unequal.

The Nodal Cubic.

9. Let us now consider separately the case of the nodal cubic.

We wish to shew that, "if ABC and DEF be two inscribed apolar tri-

angles subtending apolar pencils of lines at the node, then will ABC and DEF be co-apolar."

Let the nodal cubic be $x^3+y^3+6xyz = 0$,

i.e., $$x = 6t^2, \quad y = 6t, \quad z = -(1+t^3).$$

Then the triangle whose vertices are given by

$$t^3+3pt^2+3qt+r = 0,$$

will be an apolar triangle if

$$9t_1t_2t_3 = (t_1+t_2+t_3)(t_2t_3+t_3t_1+t_1t_2),$$

i.e., if $$r = pq.$$

The geometrical interpretation of this result is easily found to be that the tangents at the Hessian points of the triad

$$t^3+3pt^2+3qt+pq = 0,$$

intersect on the curve.

Let the inscribed apolar triangles ABC and DEF be given by

$$t^3+3pt^2+3qt+pq = 0,$$

and $$t^3+3rt^2+3st+rs = 0,$$

respectively.

Then the triangles ABC and DEF can easily be shewn to subtend apolar pencils at the point t on the curve, if

$$(pq-3qr+3ps-rs)(pqrst^3+3qst^2+3prt+1) = 0.$$

The last condition vanishes for all values of t if

$$pq-3qr+3ps-rs = 0,$$

i.e., if the triangles ABC and DEF subtend apolar pencils at the node.

10. We can now find a complete geometrical representation of the theorem of Art. 9, if we introduce the twisted cubic.

We note first of all that since the tangents at the Hessian points of an apolar triad ABC meet on the curve, their parameters will harmonically separate the parameters of the node.

Let now ABC and DEF be two triads of points on a twisted cubic, such that the planes ABC and DEF are conjugate to one another (*i.e.*, the pole of the one plane lies on the other). Let the chord through the pole of the plane ABC meet the twisted cubic in the points H and K (*i.e.*, the Hessian points of the triad ABC). Let, similarly, X and Y be the

points in which the chord through the pole of the plane DEF meets the twisted cubic. Let L and M be the double points of the involution determined by H, K and X, Y. Project the twisted cubic from any point on the chord LM, and we shall get a nodal cubic capable of being apolarly generated by means of the triangles ABC and DEF.

For, since the planes ABC and DEF are conjugate to one another, the triads ABC and DEF will be apolar to one another, and so will their projections. Also, since the twisted cubic is projected from a point on the chord LM, L' and M' will be the overlapping points of the node in the projected figure. Furthermore, H' and K' will harmonically separate L' and M', and thus $A'B'C'$ will be an apolar triangle. Similarly with regard to $D'E'F'$. Thus $A'B'C'$ and $D'E'F'$ will be inscribed apolar triangles subtending apolar pencils of lines at the node and they will therefore be co-apolar.

[I wish to express my indebtedness to Mr. J. H. Grace for his help and criticism during the preparation of this paper.]

SOME CRITERIA FOR THE RESIDUES OF EIGHTH AND OTHER POWERS

By A. E. WESTERN.

[Received August 9th, 1910.—Read April 22nd, 1909.]

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1. This paper falls into two distinct portions, Parts I–IV and Parts V–VI. The object of the first four parts is to develop some of the practical consequences of the law of l^n-th reciprocity proved in my two papers entitled "An Extension of Eisenstein's Law of Reciprocity."* The notation used in the first four parts is as follows.

$p = ml^n + 1$ is a rational prime number.

ξ is a primitive l^n-th root of 1.

The field of complex numbers defined by ξ is called $k(\xi)$.

This paper is confined to the cases $l^n = 2^3$, 2^4, and 3^2.

Rational numbers are denoted by italic letters, and complex numbers of the field $k(\xi)$ by Greek letters.

π is a prime factor of p in the field $k(\xi)$.

q is a rational prime, different from l and p.

All congruences are to mod q, unless the contrary is stated.

E is the exponent to which q belongs, mod l^n, so that

$$q^E \equiv 1 \pmod{l^n}.$$

Q is written for $(q^E - 1) l^{-n}$.

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 6, First Paper, p. 16; Second Paper, p. 265. These are hereafter cited as "first paper" and "second paper."

F is defined by the following equations:—

when $l = 2$, $\quad EF = 2^{n-2}$;

when $l \neq 2$, $\quad EF = l^{n-1}(l-1)$.

ψ_g, where $g = 1, 2, \ldots, l^n - 2$, denotes the reciprocal factors of p, so called because of their property

$$\psi_g . \psi_g(\xi^{-1}) = p.$$

In my first and second papers I have shewn how these reciprocal factors may be expressed in terms of π and its conjugates.

We know from Fermat's theorem that

$$q^{p-1} \equiv 1 \pmod{\pi},$$

and it follows from this that q^m, that is, $q^{(p-1)l^{-n}}$, is congruent, mod π, to some power of ξ. That power of ξ is denoted by $\{q/\pi\}_{l^n}$ or $\{q/\pi\}$, the suffix l^n being omitted when there can be no doubt which l^n is meant. In Parts I and II, $\{q/\pi\}$ means $\{q/\pi\}_8$; in Part III it means $\{q/\pi\}_{16}$; and in Part IV it means $\{q/\pi\}_9$.

When $\{q/\pi\} = 1$, then

$$q \equiv x^{l^n} \pmod{p};$$

and, conversely, $\{q/\pi\} = 1$ is the condition that q is the residue of the l^n-th power of some number, mod p.

If ν is composite, and its prime factors are $\pi, \pi', \ldots$, then $\{a/\nu\}$ is defined to mean $\{a/\pi\}\{a/\pi'\}\ldots$.

2. The general results proved in my first and second papers, which will be used in this paper, are as follows:—

$$\{q/\pi\}^q \equiv \psi_1 \psi_2 \ldots \psi_{q-1} \pmod{q}^*;$$

the law of reciprocity, namely,

$$\{q/\pi\} = \{\pi/q\},$$

where π is "primary" (in the sense defined in my papers) and, when $l = 2$, q is taken with such sign as to make

$$q \equiv 1 \pmod 4;\dagger$$

* First paper, § 7; and second paper, § 19.

† First paper, § 19; second paper, § 23.

when $l = 2$, and $$q \not\equiv -1 \pmod{2^n},$$

$$\{\pi/q\} \equiv \Pi \pi_u^{Qr_{-u}} \pi_u^{\dagger - Qr_{-u}} \pmod{q} \quad (u = 0, 1, \ldots, F-1),$$

wherein r denotes 5 or some power of 5 chosen to satisfy

$$q \equiv \pm r^F \pmod{2^n}.$$

r_{-u} denotes the least positive residue of $r^{-u} \pmod{2^n}$,

π_u denotes $s^u\pi$, that is, $\pi(\xi^{r^u})$,

s being the substitution which changes ξ into ξ^r, and $\pi_u^\dagger$ denotes $s^u t \pi$, that is, $\pi(\xi^{-r^u})$,

t being the substitution which changes ξ into ξ^{-1};*

and lastly, when l is odd,

$$\{\pi/q\} \equiv \Pi \pi_u^{Qr_{-u}} \pmod{q} \quad (u = 0, 1, \ldots, F-1),$$

wherein r is a primitive root, mod l^n, chosen to satisfy

$$q \equiv r^F \pmod{l^n},$$

and π_u denotes $s^u\pi$, that is, $\pi(\xi^{r^u})$,

s being the substitution which changes ξ into ξ^r.‡

3. In the first four parts of this paper, it is shewn how the general results mentioned in the previous paragraph may be simplified in the cases $l^n = 8$, 16, and 9, and how, for any given value of q, the value of $\{q/\pi\}$ may be calculated; and also the simplest expressions for $\{q/\pi\}$ are ascertained for certain small values of q. Some of these latter results have been enunciated by Bickmore§ and by Lt.-Col. A. Cunningham,‖ but it is believed that no proofs of them have hitherto been published.

In the paper just cited, Cunningham has also stated some other expressions for $\{q/\pi\}_8$, which he has discovered by induction from numerical results. In Parts V and VI the proofs of these expressions are obtained. I may refer to the first section of Part V for a general account of the subject-matter of Parts V and VI. In the theory of numbers important theorems have often (indeed, usually) been discovered by means of numerical instances, or, one may say, experimentally, and have afterwards been rigidly proved; while their existence would never have been suspected if the discoverer had not marshalled his columns of figures and observed the general law running through them. Cunningham's new expressions for $\{q/\pi\}_8$ form a remarkable instance of this tendency.

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* Second paper, § 19; and below, § 15.

‡ First paper, § 5.

§ "On the Numerical Factors of $a^n - 1$," *Messenger of Math.*, Vol. XXVI, p. 1.

‖ "On 8-vic, 16-ic, &c. Residuacity," *ante*, p. 1 of this volume.

PART I.

Criteria for the Value of $q^{\frac{1}{8}(p-1)} \pmod p$ *in Terms of the Prime Factors of* p *in the Field of* 8*-th Roots of* 1 *or of the Reciprocal Factors of* p *in that Field.*

4. It is convenient to begin with Gauss's law of quartic reciprocity for numbers in the field of 4-th roots of 1; this is included as a particular case in my second paper, Part II, and I shall deduce it from the general formulæ of §§ 19 and 23.

In the field of 4-th roots of 1, the primary prime factors of p are

$$\pi = a+b\iota \quad \text{and} \quad \pi^\dagger = a-b\iota,$$

where b is even, and $\qquad p = \pi\pi^\dagger = a^2+b^2.$

When $q = 4k+1$, $\qquad Q = \frac{1}{4}(q-1) = k,$

and so $\qquad \{q/\pi\}_4 = \{\pi/q\}_4 \equiv (\pi\pi^{\dagger -1})^k.$

And, when $q = 4k-1$, $\quad Q = \frac{1}{4}(q^2-1) = k(q-1),$

and $\qquad \pi^q = (a+b\iota)^q \equiv a+b\iota^q \equiv \pi^\dagger,$

and then $\qquad \{-q/\pi\}_4 = \{\pi/q\}_4 \equiv \pi^Q$

$$\equiv (\pi\pi^{\dagger -1})^{-k};$$

if $(q/p) = 1$, $\qquad \{-q/\pi\}_4 = \pm 1,$

and then $\qquad \{-q/\pi\}_4 \equiv (\pi\pi^{\dagger -1})^k.$

And if $p = 8m+1$, $\qquad \{-q/\pi\}_4 = \{q/\pi\}_4.$

Assuming that $(q/p) = 1$ and $p = 8m+1$, we see that the two cases $q = 4k+1$ and $4k-1$ take the same form,

$$\{q/\pi\}_4 \equiv \left(\frac{a+b\iota}{a-b\iota}\right)^k.$$

5. The criterion for $\{q/\pi\}_4 = 1$ is now easily obtained for any given value of q. Let $\qquad (a+b\iota)^k = X+\iota X',$

where X and X' are rational functions of a and b; then

$$(a-b\iota)^k = X-\iota X' \quad \text{and} \quad \{q/\pi\}_4 = 1$$

if, and only if, $\qquad X' \equiv 0.$

This is a congruence of degree k, and its solution gives as the criteria results of some of the forms

$$a \equiv 0, \qquad b \equiv 0, \qquad a \equiv \pm xb.$$

For all values of k, $b \equiv 0$ is one of the roots, and, when k is even, $a \equiv 0$ is also a root. Then $p = a^2+b^2 \equiv b^2$, a^2, or $(1+x^2)b^2$ respectively. Cunningham has calculated these criteria for all values of q up to 50.*

6. We now take the field of 8-th roots of 1; the notation to be used will be that of my second paper, § 29, namely,

$$\pi = a_0+a_1\xi+a_2\xi^2+a_3\xi^3,$$
$$\omega = \xi+\xi^3 = \sqrt{(-2)},$$
$$p = c^2+2d^2 = (c+d\omega)(c-d\omega),$$

s is here the substitution $(\xi : \xi^5)$, and t is $(\xi : \xi^{-1})$.

The values of the reciprocal factors in this field are given in my second paper, § 29, whence we obtain

$$\psi_1\psi_2 = (-1)^m(a-b\iota)(c-d\omega),$$
$$\psi_3\psi_4 = (-1)^m(c-d\omega)^2,$$
$$\psi_5\psi_6 = (-1)^m(a-b\iota)(c-d\omega),$$
$$\psi_7\psi_8 = (-1)^m p.$$

Also
$$\psi_{x+8y} = \psi_x.$$

It should be remembered that the numbers $a+b\iota$ and $c+d\omega$ are defined in terms of π, which is a given primary prime factor of p in the field. It follows from this definition that $a \equiv c \pmod 4$; a and c are therefore not necessarily positive, but possess such sign as to satisfy this congruence. In order to fix the sign, I take

$$a \equiv c \equiv 1 \pmod 4.$$

Then, using the formula of the second paper, § 19,

$$\{q/\pi\}^q \equiv \psi_1 \dots \psi_{q-1},$$

and operating on it with st when $q \equiv 3 \pmod 8$, with s when $q \equiv 5 \pmod 8$, and with t when $q \equiv 7 \pmod 8$, we obtain

when $q \equiv 1 \pmod 8$, $\quad \{q/\pi\} \equiv p^{\frac{1}{8}(q-1)}(a-b\iota)^{\frac{1}{4}(q-1)}(c-d\omega)^{\frac{1}{2}(q-1)}$,

when $q \equiv 3 \pmod 8$, $\quad \{-q/\pi\} \equiv p^{\frac{1}{8}(q-3)}(a+b\iota)^{\frac{1}{4}(q+1)}(c-d\omega)^{\frac{1}{2}(q-1)}$,

when $q \equiv 5 \pmod 8$, $\quad \{q/\pi\} \equiv p^{\frac{1}{8}(q-5)}(a-b\iota)^{\frac{1}{4}(q-1)}(c+d\omega)^{\frac{1}{2}(q+1)}$,

when $q \equiv 7 \pmod 8$, $\quad \{-q/\pi\} \equiv p^{\frac{1}{8}(q-7)}(a+b\iota)^{\frac{1}{4}(q+1)}(c+d\omega)^{\frac{1}{2}(q+1)}$.

* Unpublished.

In the particular cases of $\{q/\pi\} = \pm 1$, the signs of ι and ω in these congruences are immaterial, for the substitutions s and t can be applied without altering $\{q/\pi\}$.

Assuming that $\{q/\pi\} = \pm 1$, we have, from § 5, $X' \equiv 0$, and so

$$(a+b\iota)^k \equiv X.$$

Writing k' for the index of $c+d\omega$, viz., $\frac{1}{2}(q-1)$ or $\frac{1}{2}(q+1)$, we have

$$(c+d\omega)^{k'} \equiv Y+Y'\omega,$$

where Y and Y' are functions of c and d.

Then, since $\{q/\pi\} \equiv \pm 1$, $Y' \equiv 0$, which gives as one root $d \equiv 0$; and, when k' is even, $c \equiv 0$ is another root; and the other roots are of the form

$$c \equiv \pm yd.$$

Then $p = c^2+2d^2 \equiv c^2,\ 2d^2,\ (2+y^2)d^2$ respectively.

Finally,
$$\{\pm q/\pi\} \equiv p^j XY,$$

where j is the index of p in the above congruences; and this is readily simplified for any given value of q, taking in turn the different roots of $X' \equiv 0$ and $Y' \equiv 0$.

I give the full results for $q = 3$ and 5, and the values of $\{q/\pi\}_8$, supposing that $\{q/\pi\}_4 = 1$, for $q = 7$, 11, and 13.

7. *The case* $q = 3$. Since $p = a^2+b^2 = c^2+2d^2$, when $p \equiv 1$, either a or $b \equiv 0$, and $d \equiv 0$; when $p \equiv -1$, $a \equiv \pm b$, and $c \equiv 0$. We then obtain, from the results of the previous paragraph:—

(i) When $p \equiv 1$ and $b \equiv 0$, $\{3/\pi\} \equiv (-1)^m ac$,

(ii) When $p \equiv 1$ and $a \equiv 0$, $\{3/\pi\} \equiv (-1)^m bc\iota$,

(iii) When $p \equiv -1$ and $a \equiv -b$, $\{3/\pi\} \equiv (-1)^m ad\zeta$,

(iv) When $p \equiv -1$ and $a \equiv b$, $\{3/\pi\} \equiv (-1)^m ad\zeta^3$.

The first of these results was discovered by Bickmore* by means of numerical induction, and it can be easily identified with Cunningham's criterion.† For, if his $\alpha \equiv 0 \pmod 2$, my $a \equiv 1 \pmod 3$, and if his $\alpha \equiv 1 \pmod 2$, my $a \equiv -1 \pmod 3$; that is, $a \equiv (-1)^\alpha \pmod 3$, and similarly $c \equiv (-1)^\gamma \pmod 3$; and so in case (i), when $\{3/\pi\}_4 = 1$,

$$\{3/\pi\} = (-1)^{m+\alpha+\gamma},$$

which is Cunningham's criterion.

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* *Messenger of Math.*, Vol. XXVI, p. 15.

† *Ante*, p. 10 of this volume.

8. *The case* $q = 5$.

(i) When $p \equiv \pm 1$ and $b \equiv 0$,

(1) if $d \equiv 0$, $\quad \{5/\pi\} \equiv \quad pac$,

(2) if $d \equiv \pm 2c$, $\quad \{5/\pi\} \equiv -2pac$.

(ii) When $p \equiv \pm 1$ and $a \equiv 0$,

(1) if $d \equiv 0$, $\quad \{5/\pi\} \equiv - \ pbc\iota$,

(2) if $d \equiv \pm 2c$, $\quad \{5/\pi\} \equiv \quad 2pbc\iota$.

(iii) When $p \equiv \pm 2$ and $a \equiv b$,

(1) if $c \equiv 0$, $\quad \{5/\pi\} \equiv -2pad\xi$,

(2) if $d \equiv \pm c$, $\quad \{5/\pi\} \equiv -pad\xi$.

(iv) When $p \equiv \pm 2$ and $a \equiv -b$,

(1) if $c \equiv 0$, $\quad \{5/\pi\} \equiv -2pad\xi^3$.

(2) if $d \equiv \pm c$, $\quad \{5/\pi\} \equiv -pad\xi^3$.

The results of case (i) are due to Jacobi.* In order to deduce Cunningham's criterion from them, we need only the following:—

When $p \equiv 1$, $a \equiv (-1)^\alpha$, and $c \equiv (-1)^\gamma$ or $2(-1)^\gamma$, according as $d \equiv 0$ or not.

And, when $p \equiv -1$, $a \equiv 2(-1)^\alpha$; and $c \equiv 2(-1)^\gamma$ or $-(-1)^\gamma$, according as $d \equiv 0$ or not.

9. *The case* $q = 7$.

Assuming that $\{7/\pi\}_4 = 1$,

when $b \equiv 0$ and c or $d \equiv 0$, $\quad \{7/\pi\} = (-1)^m$,

when $b \equiv 0$ and $c \equiv \pm 3d$, $\quad \{7/\pi\} = (-1)^{m+1}$,

when $a \equiv 0$ and c or $d \equiv 0$, $\quad \{7/\pi\} = (-1)^{m+1}$,

when $a \equiv 0$ and $c \equiv \pm 3d$, $\quad \{7/\pi\} = (-1)^m$.

Cunningham's definition of α and γ in this case amount to this, that $\alpha \equiv 0 \pmod 2$ when $b \equiv 0$, and $\alpha \equiv 1 \pmod 2$ when $a \equiv 0$; that $\gamma \equiv 0 \pmod 2$ when $d \equiv 0$, and $\gamma \equiv 1 \pmod 2$ when $c \equiv \pm 3d$.

* Bickmore, *loc. cit.*, p. 15.

10. *The case* $q = 11$.

Assuming that $\{11/\pi\}_4 = 1$, and introducing for brevity the definition that $\kappa = 0$, when $d \equiv 0$ or $\pm c$, and $\kappa = 1$, when $d \equiv \pm 2c$, we obtain

$$\{11/\pi\} \equiv (-1)^{m+\kappa}(ac/11).$$

In this case, Cunningham's definitions of α and γ are, that

$$(-1)^\alpha = (a/11) \quad \text{and} \quad (-1)^\gamma = (c/11).$$

11. *The case* $q = 13$.

Assuming that $\{13/\pi\}_4 = 1$, $\{13/\pi\} = 1$ when

$$a \equiv c, \ -2c, \ -3c, \ -5c, \ 6c;$$

also, when $b \equiv 0$ and $a \equiv 4c$, and when $b \equiv \pm 4a$ and $a \equiv -4c$. And, if any of these congruences are true with the sign of c changed, then

$$\{13/\pi\} = -1.$$

12. From the value of $\{1+\iota/\pi\}_4$, one of the so-called supplementary laws in the field of 4-th roots of 1,* the value of $\{2/\pi\}$ may be easily deduced. For $2 = -\iota(1+\iota)^2$ and $b \equiv 0 \pmod 4$, and so

$$\begin{aligned}\{2/\pi\} &= \iota^{-m}\{1+\iota/\pi\}_4 \\ &= \iota^{-m+\frac{1}{4}(a-b-1)}.\end{aligned}$$

If $b \equiv 0 \pmod 8$, then $a \equiv 1-4m \pmod{16}$, and so

$$\{2/\pi\} = (-1)^{m+\frac{1}{8}b}.$$

And, if $b \equiv 4 \pmod 8$, then $a \equiv 1-4(m+2) \pmod{16}$, and so

$$\{2/\pi\} = \iota^{2m+2-\frac{1}{4}b}.$$

The former of these expressions for $\{2/\pi\}$ was stated by Reuschle, in 1856, in a slightly different form; he discovered it by induction from numerical instances.† It is remarkable that the proof of the criterion for $\{2/\pi\}$ should now be published apparently for the first time.

13. Combining the value of $\{2/\pi\}$ with that of $\{q/\pi\}$, we can easily find the value of $\{2q/\pi\}$, or, in particular, a criterion to decide whether it is ± 1. The results stated by Bickmore for $\{6/\pi\}$ and $\{10/\pi\}$ are thus

* Gauss, "Theoria Residuorum Biquad.," Comm. II (1831), *Werke*, Bd. II; and H. J. S. Smith, "Report on the Theory of Numbers," *Collected Papers*, Vol. I, p. 77.

† A. Cunningham, *Proc. London Math. Soc.*, Ser. 1, Vol. XXVII, p. 88.

verified. Similarly criteria may be derived for $\{qq'/\pi\}$, q and q' being different primes. Thus $\{15/\pi\} = 1$, when

$$\{3/\pi\} = \xi^n, \qquad \{5/\pi\} = \xi^{-n} \quad (n = 0, 1, \ldots, 7);$$

but the combinations of the different cases are too numerous to be set out. For any given value of π, it is simpler to calculate $\{q/\pi\}$ and $\{q'/\pi\}$ separately.

14. So far, the value of $\{q/\pi\}$ has been expressed in terms of the reciprocal factors of p, that is, $a+b\iota$ and $c+d\omega$. These expressions are convenient for numerical calculations, because there are extensive tables of a, b, c, and d.* Criteria for $\{q/\pi\}$ may, however, be also obtained in terms of π itself. For, if n be any prime or power of a prime, the law of reciprocity shews that the residues of π (mod q) are divisible into n sets, one of which contains the values of π such that $\{q/\pi\}_n = 1$, and the other $n-1$ of which correspond to the other values of $\{q/\pi\}_n$. A table, which for given values of n and q gives those residues of π (mod q) making $\{q/\pi\}_n = 1$ or any given power of ξ, would be a theoretically complete solution of the question. Legendre has given such a table as this for quadratic residues for all values of q not divisible by a square up to 79.‡ But in most of the cases considered in this paper, such a table would be of great size, and it must therefore suffice to state the conditions which π must satisfy in a more condensed form.

15. It should be observed that the expressions for $\{\nu/q\}$ given in my second paper, § 19, are not in general unique; when $q \equiv -1 \pmod 4$, but not $\equiv -1 \pmod{2^n}$, the factorising group§ of q, that is, the group of operations leaving $\mathfrak{q}$ (a prime factor of q) unaltered, is $\{s^F t\}$; consequently $s^F\mathfrak{q} = t\mathfrak{q}$, and the conjugates of $\mathfrak{q}$ may be written in the form

$$\mathfrak{q}_u = s^u\mathfrak{q} \quad \text{and} \quad \mathfrak{q}'_u = s^u t\mathfrak{q} \quad (u = 0, 1, \ldots, F-1).$$

In this case we obtain for $\{\nu/q\}$ the same expression as when $q \equiv 1 \pmod 4$,‖ and this will be found to be the best expression in practice. This argument does not hold when $q \equiv -1 \pmod{2^n}$, as then the factorising group is $\{t\}$.

* A. Cunningham, *Quadratic Partitions*, London, 1904.

‡ *Théorie des Nombres*, 3rd edition, Table III.

§ German, *Zerlegungsgruppe*.

‖ This is given in § 2 above.

The case $q = 3$.

Here $Q = \frac{1}{8}(3^2-1) = 1$, and

$$\{-3/\pi\} \equiv \pi\pi^{\dagger -1}.$$

Then $\{-3/\pi\} = 1$, if $\pi \equiv \pi^{\dagger}$,

which gives $a_2 \equiv 0$,

$$a_1+a_3 \equiv 0.$$

The case $q = 5$.

Here $Q = \frac{1}{8}(5^2-1) = 3$, and

$$\{5/\pi\} \equiv (\pi\pi^{\dagger -1})^3.$$

The solutions of $\kappa^3 \equiv 1$ are

$$\kappa \equiv 1,\ 2+\omega,\ 2-\omega.$$

Therefore, if $\{5/\pi\} = 1$,

$$\pi\pi^{\dagger -1} \equiv 1,\ 2+\omega,\ \text{or}\ 2-\omega \pmod{1+2\iota}.$$

The application of t shews that each of these congruences is true mod $1-2\iota$, so we get

$$\pi^{\dagger} \equiv \pi,\ (2+\omega)\pi\ \text{or}\ (2-\omega)\pi\ ;$$

and, conversely, each of these gives $\{5/\pi\} = 1$.

If $\pi^{\dagger} \equiv \pi$, then $a_2 \equiv 0$, $a_1+a_3 \equiv 0$.

If $\pi^{\dagger} \equiv (2+\omega)\pi$, then $a_0 \equiv a_1+a_3$, $2a_2 \equiv a_1-a_3$.

If $\pi^{\dagger} \equiv (2-\omega)\pi$, then $-a_0 \equiv a_1+a_3$, $-2a_2 \equiv a_1-a_3$.

The connection between these results and the criterion given before is as follows.

$$\{5/\pi\} \equiv \left(\frac{\pi}{\pi^{\dagger}}\right)^3 \equiv \frac{\pi_1\pi^{\dagger 2}}{\pi_1^{\dagger}\pi^2} \equiv \frac{\pi^{\dagger}}{\pi}\ \frac{c-d\omega}{c+d\omega};$$

so, assuming that $\{5/\pi\} = 1$,

when $\pi^{\dagger} \equiv \pi$, $d \equiv 0$,

when $\pi^{\dagger} \equiv (2+\omega)\pi$, $d \equiv 2c$,

when $\pi^{\dagger} \equiv (2-\omega)\pi$, $d \equiv -2c$.

In the first case, we get $a \equiv a_0^2-2a_1^2 \equiv c$.

In the other two cases, we get

$$a \equiv -a_0^2+2a_2^2 \equiv -2c.$$

PART II.

Criteria for the Value of $q^{\frac{1}{8}(p-1)} \pmod p$, *in Terms of* $a, b,$ *and* e, f.

16. Besides the two reciprocal factors of p in the field of 8-th roots of 1, which are defined in my second paper, § 29, there is one other way of pairing the conjugate prime factors of p, namely,

$$\pi\pi^\dagger = e+f\varpi,$$

where $$\varpi = \xi+\xi^{-1} \quad \text{and} \quad \varpi^2 = 2.$$

We then have $$p = e^2-2f^2,$$

and $$e = a_0^2+a_1^2+a_2^2+a_3^2,$$

$$f = a_0a_1+a_1a_2+a_2a_3-a_3a_0.$$

As is well known, there is a singly infinite set of values of e, f, satisfying $p = e^2-2f^2$, which correspond to the set of prime factors of p, $\epsilon^x\pi$, ϵ being the fundamental unit $1+\varpi$. Assuming that π is canonical [*i.e.*, that $\pi \equiv 1 \pmod 2$], so also is $\epsilon^{2x}\pi$, for $\epsilon^2 = 3+2\varpi \equiv 1 \pmod 2$; and therefore for the present purpose we need only consider the e and f formed from π, and from $\pi' = \epsilon\pi$. Let

$$e'+f'\varpi = \pi'\pi'^\dagger = \epsilon^2(e+f\varpi);$$

then we find $e \equiv 1 \pmod 4$, and $f \equiv a_1+a_3 \equiv d \pmod 4$;

also $e' \equiv -1 \pmod 4$, and $f' \equiv d+2 \pmod 4$.

17. The formulæ for $\{q/\pi\}$ of § 6 can be transformed so as to eliminate $c \pm d\omega$, and to introduce $e \pm f\varpi$. First, let $q = 8x+1$; then since

$$a-b\iota = \pi^\dagger\pi_1^\dagger, \quad \text{and} \quad c-d\omega = \pi_1\pi^\dagger,$$

$$\begin{aligned}\{q/\pi\} &\equiv p^x(a-b\iota)^{2x}(c-d\omega)^{4x}\\ &\equiv p^x\pi^{\dagger 6x}\pi_1^{4x}\pi_1^{\dagger 2x}\\ &\equiv p^{-x}\pi^{2x}\pi^{\dagger 8x}\pi_1^{6x}\pi_1^{\dagger 4x},\end{aligned}$$

but $$\pi^{\dagger 8x} = \pi^{\dagger q-1} \equiv 1,$$

so $$\begin{aligned}\{q/\pi\} &\equiv p^{-x}\pi^{2x}\pi_1^{6x}\pi_1^{\dagger 4x}\\ &\equiv p^{-x}(a+b\iota)^{2x}(e-f\varpi)^{4x}.\end{aligned}$$

In a similar way we find

when $q = 8x+3$, $\{-q/\pi\} \equiv p^{-x-1}(a-b\iota)^{2x+1}(e+f\varpi)^{4x+2}$,

when $q = 8x+5$, $\{\ \ q/\pi\} \equiv p^{-x-1}(a+b\iota)^{2x+1}(e+f\varpi)^{4x+3}$,

when $q = 8x+7$, $\{-q/\pi\} \equiv p^{-x-1}(a-b\iota)^{2x+2}(e-f\varpi)^{4x+3}$.

If $q \equiv \pm 1 \pmod 8$, $\epsilon^{q-1} \equiv 1$,

and if $q \equiv \pm 3 \pmod 8$, $\epsilon^q \equiv \epsilon_1$, and $\epsilon^{q+1} \equiv -1$.

When therefore $\{\pm q/\pi\}$ is expressed in terms of $e' \pm f'\varpi$, the above formulæ remain true when $q \equiv \pm 1 \pmod 8$, but the sign of the right side is changed when $q = \pm 3 \pmod 8$.

These formulæ may now be reduced for any given value of q, and thus criteria for $\{q/\pi\}$ may be found in terms of a, b, and e, f. In the rest of this Part, it is assumed that $\{q/\pi\}_4 = 1$.

18. *The case* $q = 3$.

When $f \equiv 0$, $\{-3/\pi\} \equiv a$;

when $e \equiv 0$, $\{-3/\pi\} \equiv -a$.

The case $q = 5$.

When $f \equiv 0$, $\{5/\pi\} \equiv ae$;

when $e \equiv \pm f$, $\{5/\pi\} \equiv -2ae$.

The case $q = 7$.

When $b \equiv 0$ and $f \equiv 0$, $\{-7/\pi\} = (e/7)$,

when $b \equiv 0$ and $e \equiv \pm 2f$, $\{-7/\pi\} = -(e/7)$,

when $a \equiv 0$ and $f \equiv 0$, $\{-7/\pi\} = -(e/7)$,

when $a \equiv 0$ and $e \equiv \pm 2f$, $\{-7/\pi\} = (e/7)$.

The case $q = 11$.

When $f \equiv 0$ or $f \equiv \pm 2e$, $\{-11/\pi\} = (a/11)$,

when $e \equiv 0$ or $f \equiv \pm 3e$, $\{-11/\pi\} = -(a/11)$.

I have verified Cunningham's criteria in these cases, and find them correct both for e, f, and for e', f'.

PART III.

Criteria for 3 or 5 being a Residue of a 16-th Power.

19. As might be expected, the evaluation of $\{q/\pi\}_{16}$ is more troublesome than that of $\{q/\pi\}_8$, and, except when $q = 3$, the only practicable method seems to be that already employed in §§ 14 and 15.

In this part the notation is the same as in Part III of my second paper; π is assumed to be primary; by the use of the tables of primary residues given there, any prime factor of p can be made primary by multiplying it by a suitable power of ζ. It is assumed that

$$\{q/\pi\}_8 = 1, \quad \text{so that} \quad \{q/\pi\} = \pm 1.$$

20. *The case* $q = 3$.

Using the general formula of § 19 of my second paper, and substituting in it the values of ψ_1 and ψ_2 given in Part III of that paper, we find

$$\{3/\pi\}^3 \equiv (-1)^m t\Phi_1 . t\Phi_2;$$

or, since we suppose $\{3/\pi\}$ to be ± 1,

$$\{-3/\pi\} \equiv \Phi_1\Phi_2.$$

Now $$\Phi_1 . t\Phi_1 = p \equiv 1,$$

so $$\Phi_2 \equiv \{-3/\pi\}\, t\Phi_1,$$

that is, Φ_2 (mod 3) belongs to the sub-field $(\omega_1, \omega_2, \omega_3)$; and therefore

$$d_4 \equiv 0, \quad d_2 \equiv -d_6, \quad d_3 \equiv d_5, \quad d_1 \equiv d_7,$$

and $$\Phi_2 \equiv d_0 + d_1\omega_1 + d_2\omega_2 + d_3\omega_3.$$

Therefore $$\{-3/\pi\} \equiv \frac{d_0}{c_0} \equiv -\frac{d_1}{c_1} \equiv \frac{d_2}{c_2} \equiv -\frac{d_3}{c_3}.$$

It may be proved that $c_0 \not\equiv 0$, and so we find that

$$\{-3/\pi\} \equiv c_0 d_0, \quad \text{and} \quad \{3/\pi\} \equiv (-1)^m c_0 d_0.$$

21. A criterion for $\{3/\pi\}$ may also be found in terms of π. The prime factors of 3 are $1-\omega$ and $1+\omega$, and its factorising group is $\{st\}$; also

$$Q = \tfrac{1}{16}(3^4-1) = 5,$$

and so $$\{-3/\pi\} \equiv (\pi\pi^{t-1})^5.$$

The norm of $1-\omega$ is 3^4, and so the number of incongruent residues of π prime to 3 is $(3^4-1)^2$, and the number of residues of π satisfying $\{-3/\pi\} = 1$ is one-sixteenth of this number, *i.e.*, 400.

Any residue of $1-\omega$ may be written in the form

$$\lambda \equiv x+y\omega_1+z\omega_2+w\omega_3 \pmod{1-\omega}.$$

Since s^2t leaves λ unaltered, and changes $1-\omega$ into $1+\omega$, we see that any congruence (mod $1-\omega$) in the field $(\omega_1, \omega_2, \omega_3)$ is also true mod 3.

If $\{-3/\pi\} = 1$, we must have

$$\pi^\dagger \equiv \lambda\pi \pmod{1-\omega},$$

where λ satisfies

$$\lambda^5 \equiv 1 \pmod{3}.$$

Now κ being any number of the field $k(\zeta)$,

$$\kappa^3 \equiv \kappa\,(\zeta^3) \equiv \kappa_3^\dagger;$$

therefore $\lambda^3 \equiv \lambda_3^\dagger \equiv \lambda_1$, $\lambda^4 \equiv \lambda_2$, $\lambda\lambda_2 \equiv 1$, $\lambda^2 \equiv \lambda_3$,

and the roots of $\lambda^5 \equiv 1$ are

$$1, \quad \lambda, \quad \lambda_1, \quad \lambda_2, \quad \lambda_3.$$

By trial, values of x, y, z, and w are found which satisfy $\lambda^2 \equiv \lambda_3$, and thus we get

$$\lambda \equiv -1-\omega_1-\omega_2+\omega_3.$$

Now, since $\pi^\dagger \equiv \lambda\pi \pmod{1-\omega}$,

applying t, $\pi \equiv \lambda_2\pi^\dagger \pmod{1+\omega}$,

that is, $\pi^\dagger \equiv \lambda_2^{-1}\pi \equiv \lambda\pi \pmod{1+\omega}$,

and therefore $\pi^\dagger \equiv \lambda\pi \pmod{3}$.

Therefore the complete solution of $\{-3/\pi\} = 1$ is

$$\pi^\dagger \equiv \pi, \quad \lambda\pi, \quad \lambda_1\pi, \quad \lambda_2\pi, \quad \text{or} \quad \lambda_3\pi \pmod{3};$$

and when $m \equiv 1 \pmod{2}$, $\{3/\pi\} = -\{-3/\pi\}$,

and then the solution of $\{3/\pi\} = 1$ is

$$-\pi^\dagger \equiv \pi, \quad \lambda\pi, \quad \lambda_1\pi, \quad \lambda_2\pi, \quad \text{or} \quad \lambda_3\pi.$$

Each value of $\pi\pi^{\dagger\,-1}$ corresponds to 3^4-1 values of π; for instance, $\pi \equiv \pi^\dagger$ gives

$$\pi \equiv b_0+b_1\varpi_1+b_2\varpi_2+b_3\varpi_3,$$

where b_0, b_1, ... may have any values.

22. Yet another criterion for $\{-3/\pi\}$ may be derived from the results of the last paragraph; for, if

$$\pi^\dagger \equiv \lambda^x \pi,$$

then $$\Phi_2 \equiv \pi\pi_1\pi_2\pi_3\lambda_1^x,$$

but $$\pi\pi_1\pi_2\pi_3 = a+b\iota \equiv a,$$

so $$\Phi_2 \equiv a\lambda^{3x};$$

and conversely, if $\Phi_2 \equiv a\lambda^u$, and $\{3/\pi\}_4 = 1$,

then $$\pi^\dagger \equiv \lambda^{2u}\pi,$$

and so $$\{-3/\pi\} = 1;$$

and if $\Phi_2 \equiv -a\lambda^u$, $\{-3/\pi\} = -1$.

23. *The case* $q = 5$.

Here $$Q = \tfrac{1}{16}(5^4-1) = 39,$$

and $$\{5/\pi\} \equiv (\pi\pi^{\dagger -1})^{39}.$$

As in the case of $q = 3$, any residue mod $1+2\iota$ may be written in the form

$$x+y\omega_1+z\omega_2+w\omega_3,$$

and it only remains to find the solutions of

$$\kappa^{39} \equiv 1.$$

Now, if $\lambda^{13} \equiv 1$, and $\mu^3 \equiv 1$,

$$\kappa \equiv \lambda^x\mu^y \quad (x = 0, \ldots, 12;\ y = 0, 1, 2).$$

Then $$\lambda^5 \equiv \lambda_1, \quad \lambda^{25} \equiv \lambda_2, \quad \lambda^{125} \equiv \lambda_3,$$

and so $$\lambda\lambda_2 \equiv 1, \quad \text{and} \quad \lambda^3 \equiv \lambda_3^2.$$

A value of λ satisfying the latter congruence is found by trial to be

$$\lambda \equiv \quad 1+\omega_1-2\omega_2,$$

whence $$\lambda^2 \equiv \quad 2+\omega_1+2\omega_2+\omega_3,$$

and $$\lambda^4 \equiv -2-\omega_1+\omega_2+2\omega_3;$$

and the other roots, besides 1, are λ_u, λ_u^2, λ_u^4 $(u = 1, 2, 3)$.

We have seen in § 15 that a root of $\mu^3 \equiv 1$ is $2+\omega$; for the present purpose it is better to take μ in the same field as λ, and we find

$$\mu \equiv 1, \quad 2+2\omega_2, \quad 2-2\omega_2.$$

Finally, as in the case of $q = 3$, we find that

$$\{5/\pi\} = 1,$$

when $$\pi^\dagger \equiv \lambda^x \mu^y \pi.$$

Also, as before, $$\{5/\pi\} = 1,$$

when $$\Phi_2 \equiv a\lambda^x\mu^y, \quad \text{and} \quad \{5/\pi\}_4 = 1,$$

and $$\{5/\pi\} = -1,$$

when $$\Phi_2 \equiv -a\lambda^x\mu^y, \quad \text{and} \quad \{5/\pi\}_4 = 1.$$

The curious result follows from this, that whenever $\{5/\pi\}_8 = 1$, $\Phi_2 \pmod 5$ belongs to the sub-field $(\omega_1, \omega_2, \omega_3)$.

24. Throughout this part, it should be observed that $a+b\iota$, Φ_1 and Φ_2 are defined as products of π and its conjugates; for instance,

$$a+b\iota = \pi\pi_1\pi_2\pi_3.$$

The result of substituting $\epsilon_1\pi$ for π in any of these products is to multiply it by the norm of ϵ_1 in the field of $k(\zeta+\zeta^{-1})$, that is, by -1. Accordingly if π is in one of the eight primary forms $\{\epsilon_1^2, \epsilon_3, \gamma\}$,* we have

$$a \equiv 1 \pmod 8, \quad c_0 \equiv 1 \pmod 8, \quad \text{and} \quad d_0 \equiv 1+\tfrac{1}{2}b \pmod 4,$$

as in §§ 32 and 33 of my second paper; but if π is in one of the other eight primary forms,* then

$$a \equiv -1 \pmod 8, \quad c_0 \equiv -1 \pmod 8, \quad \text{and} \quad d_0 \equiv -1-\tfrac{1}{2}b \pmod 4.$$

It will be seen that the only formulæ of this part that are affected by this are those of the last two sections containing Φ_2.

* Tables of these are given in my second paper, §§ 31 and 36.

PART IV.

Criteria for 2 or 5 being a Residue of a 9-th Power.

25. In this part, the notation is the same as in Part IV of my second paper. It is assumed that π is primary, and that $\{q/\pi\}_3 = 1$, so that $\{q/\pi\} = 1$, ρ, or ρ^2.

The case $q = 2$.

As before, the general formulæ of my second paper give

$$\{2/\pi\} \equiv \phi,$$

and also

$$\{2/\pi\} \equiv \pi_3 \pi^{-1}.$$

We infer that for every π, $\quad \phi \equiv \xi^x \pmod 2$.

The condition that $\{2/\pi\}_3 = 1$ is $a' \equiv 0$, and hence the interesting connection between $\pi^* = a + a'\rho$ and ϕ, namely, that when $a' \equiv 0$, then

$$c_1 \equiv c_2 \equiv c_7 \equiv c_8 \equiv 0.$$

We also see that the criterion for $\{2/\pi\} = 1$ is that

$$c_3 \equiv 0,$$

or that

$$\pi \equiv \pi_3.$$

The latter criterion gives

$$b_3 \equiv 0, \quad b_1 \equiv b_8, \quad b_2 \equiv b_7,$$

and so

$$\pi \equiv b_0 + b_1(\xi + \xi^{-1}) + b_2(\xi^2 + \xi^{-2}).$$

26. *The case* $q = 5$.

5 is a prime in the field of 9-th roots of 1.

$\pi^* = \pi\pi_2\pi_4 = a + a'\rho$ being a factor of p in the field of third roots of 1, the condition that $\{5/\pi\}_3 = 1$ is known to be :—

when $p \equiv \pm 1$, $\quad \pi^* \equiv \quad \pi_1^*$, that is, $\quad a' \equiv 0$;

when $p \equiv \pm 2$, $\quad \pi^* \equiv -\pi_1^*$, that is, $\quad 2a - a' \equiv 0$.

Now $\pi^5 \equiv \pi_5$, $\quad \pi^{5^2} \equiv \pi_4$, and generally $\pi^5 \equiv \pi_{6-n}$,

and so from $$\pi^* \equiv \pm \pi_1^*,$$

that is, $$\pi\pi_2\pi_4 \equiv \pm \pi_1\pi_3\pi_5,$$

we deduce $$\pi^{5^4+5^2+1} \equiv \pm \pi_3^{5^4+5^2+1}.$$

The reciprocity law gives at once

$$\{5/\pi\} = \{\pi/5\} \equiv \pi^{\frac{1}{8}(5^6-1)}$$
$$\equiv \pi^{14(5^3-1)}$$
$$\equiv (\pi_3\pi^{-1})^{14}.$$

So if $\{5/\pi\} = 1$, $$\pi^{14} \equiv \pi_3^{14}.$$

Since $$5^4+5^2+1 = 3.7.31,$$

we find that when $p \equiv \pm 1$, $\{5/\pi\} = 1$, when $\pi^7 \equiv \pi_3^7$,

and that when $p \equiv \pm 2$, $\{5/\pi\} = 1$, when $\pi^7 \equiv -\pi_3^7$.

We have now only to solve $\lambda^7 \equiv 1$.

The simplest equivalent congruence is $\lambda^2 \equiv \lambda_2$, and this can be solved by trial. The table at the end of this part gives the six solutions of $\lambda^7 \equiv 1$ other than 1.

As in Part III, the criterion for $\{5/\pi\} = 1$ may be expressed in terms of the reciprocal factor

$$\phi = \pi\pi_1\pi_2.$$

First, when $p \equiv \pm 1$, $$\pi^* \equiv a,$$

that is, $$\pi_4\phi \equiv a\pi_1,$$

and so $$\phi \equiv a\lambda_x.$$

Secondly, when $p \equiv \pm 2$, $\pi^* \equiv -\pi_1^*$,

that is, $$\pi_4^2\phi \equiv -\pi_1^2\phi_3,$$

or $$\pi_4^2\phi^2 \equiv -\pi_1^2 p.$$

Now in this case

$$p \equiv -2a^2 \quad \text{and} \quad (1+2\rho)^2 \equiv 2,$$

and so $$\phi \equiv \pm(1+2\rho)\, a\lambda_x.$$

The Solutions of $\lambda^7 \equiv 1 \pmod 5$.

	1	ζ	ζ^2	ζ^3	ζ^7	ζ^8
λ	2	1	−1	1	−2	0
$\lambda_1 \equiv \lambda^3$	1	1	−2	−1	1	2
$\lambda_2 \equiv \lambda^2$	2	2	0	1	−1	−1
$\lambda_3 \equiv \lambda^6$	1	0	−2	−1	−1	1
$\lambda_4 \equiv \lambda^4$	2	2	1	1	−2	1
$\lambda_5 \equiv \lambda^5$	1	−1	−1	−1	0	2

PART V.

The Quartic Field $k(\omega, \theta)$, *where*

$$\omega = \sqrt{(-2)}, \quad \textit{and} \quad \theta = \tfrac{1}{2}\{-1+\sqrt{[(-1)^{\frac{1}{2}(q-1)}q]}\}.$$

27. Col. A. Cunningham has given in his paper* expressions for $\{q/\pi\}_8$ in terms of a, b, t, u, t', u', where

$$p = a^2+b^2 = t^2+qu^2 = t'^2-qu'^2,$$

and it is assumed that $\{q/\pi\} = 1$. When $q = 11$,

$$p = \tfrac{1}{4}(t^2+11u^2).$$

The object of Parts V and VI of this paper is to furnish proofs of these expressions. As $\{q/\pi\}$ has already been expressed in terms of a, b and c, d, these new results depend on certain congruences between a, b, c, d, t, u, t', u'. These congruences will be proved by the use of certain quartic fields of algebraic numbers, containing as sub-fields two or three of the quadratic fields $k(\iota)$, $k[\sqrt{(-2)}]$, $k(\sqrt{q})$, $k[\sqrt{(-q)}]$. Quartic fields containing a quadratic field have been studied by Hilbert† and Sommer,‡ and are called by Hilbert "Dirichlet's Biquadratic Fields," because Dirichlet was the first to investigate the theory of binary quadratic forms whose coefficients and variables belong to the field $k(\iota)$.

28. In this part I deal with the quartic field which contains $\omega = \sqrt{(-2)}$, and $\sqrt{(eq)}$, where $e = (-1)^{\frac{1}{2}(q-1)}$. A basis of the quadratic field defined

* *Ante*, p. 10 of this volume.

† *Bericht, Deutschen Math. Verein*, Bd. 4, 1897, § 87; and *Math. Annalen*, Bd. 45, p. 309.

‡ *Vorlesungen über Zahlentheorie*, 1907.

by ω is $(1, \omega)$, and its discriminant is 8; a basis of the quadratic field defined by $\sqrt{(eq)}$ is $(1, \theta)$, where

$$\theta = \tfrac{1}{2}[-1+\sqrt{(eq)}],$$

and its discriminant is eq.

I therefore call this quartic field $k(\omega, \theta)$.

Since these discriminants are prime to each other, the discriminant of $k(\omega, \theta)$ is 2^6q^2,* and a basis is

$$(1, \omega, \theta, \omega\theta).$$

Therefore every integer of $k(\omega, \theta)$ is of the form

$$d_0+d_1\omega+d_2\theta+d_3\omega\theta,$$

where d_0, d_1, d_2, d_3 are rational integers.

The field $k(\omega, \theta)$ is clearly a sub-field of the field of $8q$-th roots of 1, and this fact furnishes another proof that $(1, \omega, \theta, \omega\theta)$ is a basis of $k(\omega, \theta)$.

θ_1 means $\tfrac{1}{2}[-1-\sqrt{(eq)}]$,

k means $\tfrac{1}{4}(1-eq)$.

Then θ and θ_1 are the roots of $\theta^2+\theta+k = 0$.

$f(x, y)$ means $(x+\theta y)(x+\theta_1 y) = x^2-xy+ky^2 = \tfrac{1}{4}[(2x-y)^2-eqy^2]$.

Then $f(x, y) \equiv 0$, or a quadratic residue mod q.

$\mathfrak{p}$ or π denotes a prime factor of p in the field $k(\omega, \theta)$.

r is the substitution $(\theta : \theta_1)$.

s is the substitution $(\omega : -\omega)$.

ξ is a primitive 8-th root of 1, and ρ a primitive q-th root of 1.

29. In a similar way to that used by Hilbert for the field $k(\iota, \theta)$,† we find that the rules of factorisation in $k(\omega, \theta)$ are as follows.

If p is an odd prime other than q,

if $p \equiv 1$ or 3 (mod 8), and $(p/q) = 1$; then the weight‡ of $\mathfrak{p}$ is 1, that is, p has four conjugate prime factors;

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* Hilbert, *Bericht*, p. 267.

† *Math. Annalen*, Bd. 45.

‡ German, *Grad*.

if $p \equiv 1$ or 3 (mod 8), and $(p/q) = -1$, the factorising group is $\{r\}$;

if $p \equiv 5$ or 7 (mod 8), and $(p/q) = 1$, the factorising group is $\{s\}$:

if $p \equiv 5$ or 7 (mod 8), and $(p/q) = -1$, the factorising group is $\{rs\}$;

in the last three cases, the weight of $\mathfrak{p}$ is 2.

When $q \equiv \pm 1$ (mod 8), a prime factor of 2 is (ω, θ), of weight 1.

When $q \equiv \pm 3$ (mod 8), a prime factor of 2 is ω, of weight 2.

When $q \equiv 1$ or 3 (mod 8), a prime factor of q is $(\kappa, \surd(eq))$, of weight 1, κ being a factor of q in the field $k(\omega)$.

When $q \equiv 5$ or 7 (mod 8), a prime factor of q is $\surd(eq)$, of weight 2.

30. The next problem is to find the class-number of the field $k(\omega, \theta)$ for the values of q with which we are concerned. For this purpose I use a theorem of Minkowski,* from which is derived this consequence:—

In any algebraic field of order n greater than 2, v being the number of pairs of conjugate imaginary fields, and d being the discriminant of the field, an ideal $\mathfrak{a}$ exists in each ideal class of the field such that

$$N(\mathfrak{a}) < \left(\frac{4}{\pi}\right)^{v} \frac{(n-1)!}{n^{n-1}} \mid d \mid^{\frac{1}{2}}.$$

In the field $k(\omega, \theta)$, $n = 4$, $v = 2$, and so

$$N(\mathfrak{a}) < 3(2\pi^2)^{-1} \mid d \mid^{\frac{1}{2}}$$

$$< q \times 1.22.$$

So when $q = 3,\ 5,\ 7,\ 11,\ 13,$

$N(\mathfrak{a}) \leqslant 3,\ 6,\ 8,\ 13,\ 15,$

and $[N(\mathfrak{a})]^{\frac{1}{2}} \leqslant 1,\ 2,\ 2,\ 3,\ 3.$

We now factorise all rational primes whose factors are of weight 1, not exceeding the limit of $N(\mathfrak{a})$; and if p has as factor a prime ideal $\mathfrak{p}$ of weight 2, then the norm of $\mathfrak{p}$ is p^2, and so in this case, only those primes p need to be factorised which do not exceed $[N(\mathfrak{a})]^{\frac{1}{2}}$.

The prime factors of all rational primes within these limits are given in the table at the end of this part. This table shews that when

$$q = 3,\ 5,\ 7,\ \text{or}\ 11,$$

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* *Geometrie der Zahlen*, pp. 122 and 134.

all these prime factors are principal ideals,* that is, actual numbers, and therefore the class-number is 1.

In the case $q = 13$, the prime factor $\mathfrak{p}$ of 3 is not a principal ideal, for, if it were, $\mathfrak{p}.rs\mathfrak{p}$ would be a number of the form $x+y\sqrt{(-26)}$, but 3 clearly cannot have such a factor. Also $\mathfrak{p}.r\mathfrak{p} = (1+\omega)$, and $\mathfrak{p}.s\mathfrak{p} = (\theta)$, and so $r\mathfrak{p}$ and $s\mathfrak{p}$ belong to the same class, and $\mathfrak{p}$ and $rs\mathfrak{p}$ belong to the same class. Again $\mathfrak{p}$ and $s\mathfrak{p}$ belong to different classes, for, if not, $\mathfrak{p}^2$ would be a principal ideal, and then would $x^2+26y^2 = 9$ be soluble. Therefore there are three classes, whose representative ideals are 1, $\mathfrak{p} = (1-\omega, \theta)$, and $s\mathfrak{p} = (1+\omega, \theta)$.

31. In the field $k(\omega, \theta)$, Dirichlet's theorem† shews that there is a single fundamental unit. For each value of q, which is $\equiv -1 \pmod 4$, I have calculated a unit ϵ of $k(\omega, \theta)$ by multiplying the cyclotomic unit $1-\xi\rho$ by some of its conjugates.

In each case in the table, $\epsilon.rs\epsilon$ is found to be the fundamental unit of $k[\sqrt{(-2eq)}]$, and therefore ϵ is the fundamental unit of $k(\omega, \theta)$.

When $q \equiv 1 \pmod 4$, it is readily proved that the cyclotomic unit $E = \Pi(1-\xi\rho)$ is a cyclotomic unit of $k(\theta)$. And in the cases $q = 5$ and 13, it may be proved that no unit ϵ exists such that $\epsilon.s\epsilon = \pm E$; therefore in these cases E is the fundamental unit of $k(\omega, \theta)$.

32. We now consider the factors of p in the fields $k(\omega)$ and $k(\theta)$, formed from

$$\pi = d_0+d_1\omega+d_2\theta+d_3\omega\theta.$$

These are
$$c'+d'\omega = \pi.r\pi,$$

and
$$A_0+A_1\theta = \pi.s\pi,$$

where
$$c = f(d_0, d_2)-2f(d_1, d_3),$$

$$d' = 2d_0d_1-d_0d_3-d_1d_2+2kd_2d_3,$$

and
$$A_0 = d_0^2+2d_1^2-k(d_2^2+2d_3^2),$$

$$A_1 = 2d_0d_2+4d_1d_3-d_2^2-2d_3^2.$$

Now
$$A_0+A_1\theta = x+y\sqrt{(eq)},$$

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* German, *Hauptideal*.

† Hilbert, *Bericht*, p. 214, Satz 47.

where $$x = A_0 - \tfrac{1}{2}A_1,$$

$$y = \tfrac{1}{2}A_1,$$

and $$x^2 - eqy^2 = p.$$

In Cunningham's notation, when $e = 1$, $x = \pm t'$, $y = \pm u'$, and when $e = -1$, $x = \pm t$, $y = \pm u$, except that when $q = 11$,

$$x = \pm \tfrac{1}{2}t, \quad y = \pm \tfrac{1}{2}u.$$

It will be observed that c' and d' may differ in sign from c and d as defined in Part I.

Then $$2x = 2f(d_0, d_2) + eqd_2^2 + 4f(d_1, d_3) + 2eqd_3^2,$$

and so $$x \equiv f(d_0, d_2) + 2f(d_1, d_3).$$

When $q \equiv 1 \pmod 4$, x is essentially positive.

When x is an integer, since $x^2 - eqy^2 = p$ and $eq \equiv 1 \pmod 4$, it follows that x is odd, and $y \equiv 0 \pmod 4$.

Then $$x^2 = (A_0 - y)^2 \equiv A_0^2 + 2y \pmod{16}.$$

And so $$1 + 8m = p \equiv x^2 \equiv A_0^2 + 2y \pmod{16}.$$

For brevity, I write $f = f(d_0, d_2)$ and $f' = f(d_1, d_3)$.

For the remainder of the work, each value of q must be treated separately; the cases $q = 3$ and 5 are dealt with below; and the processes for $q = 7$ and 11 are so similar that they are omitted. In each case, I find Cunningham's corresponding criterion to be correct.

33. *The case* $q = 3$.

Here we have $$c' \equiv f + f',$$

$$x \equiv f - f',$$

and f and f' are $\equiv 0$ or 1.

Since c' and x are both prime to 3, it follows that $c' \equiv 1$.

But $c \equiv (-1)^\gamma$, as in § 7, *supra*. So

$$c' = (-1)^\gamma c \equiv (-1)^\gamma \equiv 1 + 2\gamma \pmod 4.$$

Since $$\pi \equiv d_0 + d_2\theta \pmod \omega,$$

and θ is in this case a unit, by multiplying π by a suitable power of θ we get π in a form such that

$$d_0 \equiv 1, \quad d_2 \equiv 0 \pmod 2.$$

Since $d' \equiv 0 \pmod 2$,

we must then have $d_3 \equiv 0 \pmod 2$.

If $d_1 \equiv 1 \pmod 2$,

then $\pi \equiv 1+\omega \pmod 2$.

But the unit $\epsilon \equiv 1+\omega \pmod 2$,

and so $\epsilon\pi \equiv 1 \pmod 2$.

We may therefore take π in a form such that

$$d_0 \equiv 1, \quad d_1 \equiv d_2 \equiv d_3 \equiv 0 \pmod 2.$$

Then $c' \equiv 1+d_2 \pmod 4$,

and so $d_2 \equiv 2\gamma \pmod 4$.

Again, $A_0 \equiv 1+d_2^2 \pmod 8$,

$$\equiv 1+2d_2 \pmod 8,$$

$$\equiv 1+4\gamma \pmod 8.$$

So $A_0^2 \equiv 1+8\gamma \pmod{16}$.

Then, since $y \equiv 4\nu \pmod 8$,

$$1+8m \equiv A_0^2+2y \pmod{16}$$

$$\equiv 1+8\gamma+8\nu \pmod{16},$$

that is, $m+\gamma+\nu \equiv 0 \pmod 2$.

And so $q^{\frac{1}{8}(p-1)} \equiv (-1)^{\alpha+\gamma+m} \equiv (-1)^{\alpha+\nu} \pmod p$,

which is Cunningham's criterion.

34. *The case* $q = 5$.

Here $$\left.\begin{aligned} c' &\equiv f-2f' \\ x &\equiv f+2f', \end{aligned}\right\}$$

and f and f' are $\equiv 0$ or ± 1.

Then we get the four cases—

(i) $f' \equiv 0$, then $c' \equiv x \equiv \pm 1$, $p \equiv 1$, $d' \equiv 0$.

(ii) $f \equiv 0$, then $c' \equiv -x \equiv \pm 2$, $p \equiv -1$, $d' \equiv 0$.

(iii) $f \equiv f'$, then $c' \equiv -2x$, $p \equiv -1$, $d' \not\equiv 0$.

(iv) $f \equiv -f'$, then $c' \equiv 2x$, $p \equiv 1$, $d' \not\equiv 0$

Since in this case x is positive, it follows from Cunningham's definition of τ' and his condition* for $(5/p)_4 = 1$, that

when $p \equiv 1$, $\quad x \equiv (-1)^{\tau'}$,

and when $p \equiv -1$, $\quad x \equiv -2(-1)^{\tau'}$.

The only unit in this case being θ, we can get π (by multiplying it by θ or θ^2) in a form such that

$$d_0 \equiv 1, \quad d_1 \equiv 0 \text{ or } 1, \quad d_2 \equiv d_3 \equiv 0 \pmod 2.$$

Then $\quad c' \equiv 1+2d_1+d_2 \pmod 4$,

so $\quad c' = (-1)^{d_1+\frac{1}{2}d_2} c$.

The values of c in terms of γ are given in § 8 *supra*.

In case (i) we have $\quad c' \equiv (-1)^{\gamma+d_1+\frac{1}{2}d_2}, \quad x \equiv (-1)^{\tau'}$,

so since $\quad c' \equiv x$,

$$\gamma+\tau' \equiv d_1+\tfrac{1}{2}d_2 \pmod 2.$$

And the same result is found in the other three cases.

Now $\quad A_0 \equiv 1+2d_1^2+2d_2 \pmod 8$.

So, if $d_1 \equiv 0 \pmod 2$, $\quad A_0 \equiv 1+2d_2 \pmod 8$,

and $\quad A_0^2 \equiv 1+4d_2 \pmod{16}$;

and if $d_1 \equiv 1 \pmod 2$, $\quad A_0 \equiv 3+2d_2 \pmod 8$,

and $\quad A_0^2 \equiv 9+4d_2 \pmod{16}$.

So in each case $\quad A_0^2 \equiv 1+8d_1+4d_2 \pmod{16}$,

and then $\quad m+\nu' \equiv \tfrac{1}{8}(A_0^2-1) \equiv d_1+\tfrac{1}{2}d_2 \pmod 2$.

Therefore, finally, $\quad m+\gamma+\tau'+\nu' \equiv 0 \pmod 2$,

which proves that $\quad q^{\frac{1}{8}(p-1)} \equiv (-1)^{m+\alpha+\tau'+\nu'} \pmod p$,

which is Cunningham's criterion.

It will now be shewn that this is true, taking any solution t', u', of

$$x^2-5y^2 = p.$$

Since $\theta^3 = -2+\sqrt{5}$, $\theta^6 = 9-4\sqrt{5}$ gives the fundamental solution of

$$x^2-5y^2 = 1 ;$$

* *Proc. London Math. Soc.*, Ser. 2, Vol. 1, p. 132.

and therefore by multiplying π by a suitable power of θ^3, we can obtain π in that form which corresponds to the fundamental or any other given solution of $x^2-5y^2=p$.

Table of the Fields $k(\omega, \theta)$.

q	p	Weight of $\mathfrak{p}$	$\mathfrak{p}$	Class Number.	Fundamental Unit.
3	3	1	$1-\omega\theta$	1	$1+\omega+2\theta$
5	2	2	ω	1	θ
7	2	1	$\omega+\theta$	1	$1+2\omega+2\theta$
11	2	2	ω	1	$3+7\omega+6\theta$
	3	1	$1-\omega+\theta$		
	11	1	$3-2\omega+2\theta+\omega\theta$		
13	2	2	ω	3	$2+\theta$
	3	1	$(1-\omega, \theta)$		

PART VI.

The Quartic Field $k(\iota, \theta)$.

35. In this part, θ, r and $f(x, y)$ have the same meanings as in Part V. The field $k(\iota, \theta)$ is a sub-field of the field of $4q$-th roots of 1. A basis of the field is 1, ι, θ, $\iota\theta$, and its discriminant is 2^4q^2.

$$\pi = b_0+b_1\iota+b_2\theta+b_3\iota\theta$$

denotes a prime factor of p in the field.

λ denotes $1+\iota$.

t is the substitution $(\iota, -\iota)$.

As in Part III, an ideal $\mathfrak{d}$ exists in each ideal class such that

$$N(\mathfrak{d}) < q\times\cdot 61.$$

So when $q = 3,\ 5,\ 7,\ 11,\ 13,$

$$N(\mathfrak{d}) \leqslant 1,\ 3,\ 4,\ \ 6,\ \ 7.$$

The table at the end of this part shews that for all these values of q the class number of $k(\iota, \theta)$ is 1. The fundamental units have been calculated as in § 31.

36. The factors of p in the three quadratic sub-fields $k(\iota)$, $k(\sqrt{q})$, and $k[\sqrt{(-q)}]$, formed from π are

$$a'+b'\iota = \pi.r\pi,$$

$$A_0+A_1\theta = \pi.t\pi,$$

and $$x'+y'\sqrt{(-eq)} = \pi.rt\pi,$$

where
$$a' = f(b_0, b_2)-f(b_1, b_3),$$

$$b' = 2b_0b_1-b_0b_3-b_1b_2+2kb_2b_3,$$

$$A_0 = b_0^2+b_1^2-k(b_2^2+b_3^2),$$

$$A_1 = 2b_0b_2+2b_1b_3-b_2^2-b_3^2,$$

$$x' = f(b_0, b_2)+f(b_1, b_3),$$

$$y' = b_0b_3-b_1b_2.$$

As in § 32, $$x = A_0-\tfrac{1}{2}A_1, \quad y = \tfrac{1}{2}A_1,$$

when $e = 1$, $\quad x = \pm t', \quad y = \pm u', \quad x' = \pm t, \quad y' = \pm u,$

and when $e = -1$, $\quad x = \pm t, \quad y = \pm u, \quad x' = \pm t', \quad y' = \pm u$

(with the same exception as in § 32 for $q = 11$). And as in § 32,

$$m+\tfrac{1}{4}y \equiv \tfrac{1}{8}(A_0^2-1) \pmod 2,$$

whenever $$y \equiv 0 \pmod 4.$$

It will suffice to give the full proof of Cunningham's criterion for $q = 3$. I have verified his criteria in the cases $q = 5$, 7, and 11.

37. *The case* $q = 3$.

Since $$\pi \equiv b_0+b_1+(b_2+b_3)\theta \pmod \lambda,$$

and 1, θ and θ^2 is a complete set of residues (mod λ), by multiplying π by a suitable power of the unit θ, we get

$$b_0+b_1 \equiv 1, \quad b_2+b_3 \equiv 0 \pmod 2;$$

and then using a suitable power of ι, we get

$$b_0 \equiv 1, \quad b_1 \equiv 0 \pmod 2.$$

If $$b_2 \equiv b_3 \equiv 1 \pmod 2,$$

then $$\pi \equiv 1+\theta(1+\iota) \equiv \epsilon \pmod 2,$$

and then $$\epsilon\pi \equiv \epsilon^2 \equiv 1 \pmod 2.$$

Therefore, in the first place, we take

$$b_0 \equiv 1, \quad b_1 \equiv b_2 \equiv b_3 \equiv 0 \pmod 2.$$

Then $$b' \equiv 0 \pmod 2,$$

and so $$b' = \pm b \equiv 0 \pmod 4,$$

and therefore $$b_3 \equiv 0 \pmod 4,$$

and $$b' \equiv 2b_1+b_3+b_1b_2 \pmod 8.$$

Also $$y' \equiv b_3+b_1b_2 \pmod 8.$$

Therefore $$b'+y' \equiv 2b_1 \pmod 8,$$

that is, $$\beta+\upsilon' \equiv \tfrac{1}{2}b_1 \pmod 2.$$

x' is positive, and so $$t' = x' \equiv 1+b_2 \pmod 4,$$

and $$\tau' \equiv \tfrac{1}{2}(t'-1) \equiv \tfrac{1}{2}b_2 \pmod 2.$$

Therefore $$\beta+\tau'+\upsilon' \equiv \tfrac{1}{2}(b_1+b_2) \pmod 2.$$

Also $y \equiv 0 \pmod 4$, and $A_0 \equiv 1+2(b_1+b_2) \pmod 8$,

and so $$m+\upsilon \equiv \tfrac{1}{2}(b_1+b_2) \pmod 2,$$

and, finally, $$m+\beta+\upsilon+\tau'+\upsilon' \equiv 0 \pmod 2,$$

which proves the criterion for the case of $u' \equiv 0 \pmod 2$.* Since π may be multiplied by any power of ϵ^2 without affecting the argument, the criterion holds for any solution of $t'^2-3u'^2 = p$ in which $u' \equiv 0 \pmod 2$.

For the second case, in which $t' \equiv 0 \pmod 2$, we take $\epsilon\pi$ instead of π, and corresponding to this, we get

$$T'+U'\sqrt{3} = \epsilon.rt\epsilon(t'+u'\sqrt{3}) = (2-\sqrt{3})(t'+u'\sqrt{3}),$$

and so $$T' = 2t'-3u'.$$

Defining τ_0' by $T' = 4\tau_0'+2$,†

$$\tau_0' \equiv \tfrac{1}{2}(t'-1)+\tfrac{1}{4}u' \pmod 2$$
$$\equiv \tau'+\upsilon' \pmod 2.$$

Therefore in this case the desired congruence is

$$m+\beta+\upsilon+\tau_0' \equiv 0 \pmod 2,$$

which proves the criterion for the case of $t' \equiv 0 \pmod 2$.*

* *Ante*, p. 10 of this volume, Table II, last column.

† Cunningham writes τ' for τ_0'.

Table of the Fields $k(\iota, \theta)$.

q	p	Weight of $\mathfrak{p}$	$\mathfrak{p}$	Class Number.	Fundamental Unit.
3	—	—	—	1	$1+\theta+\iota\theta$
5	—	—	—	1	θ
7	2	1	$1+\iota+\theta$	1	$1+2\iota-\theta+\iota\theta$
11	2	2	$1+\iota$	1	$1+2\iota-\theta+\iota\theta$
	5	1	$1+\iota+\theta$		
13	2	2	$1+\iota$	1	$2+\theta$

THE COMPOSITION OF FINITE DISPLACEMENTS AND THE USE OF AXODES

By G. T. BENNETT.

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1. *The Resultant of Two Finite Displacements.*

The law of composition seems to have been first given by Halphen,* and subsequently and independently by Burnside† and Wiener.‡ It is usually stated somewhat thus :—Let a screw displacement of a body take place on an axis a, followed by a screw displacement on an axis b. Take ν the common normal of a and b. Draw a line μ, normal to a, and with distance and inclination relative to ν equal to half the shift and half the turn of the screw movement on a; the sense of the angle and distance being such that half the screw movement would bring μ to coincide with ν. Similarly take a line λ, normal to b, and with distance and inclination relative to ν equal to half the shift and half the turn of the screw movement on b. Take then the common normal of λ and μ, namely c. The resultant screw movement is a screw movement on c, with a shift and turn equal to twice the mutual distance and inclination of λ and μ.

The use of half-turns makes the statement clearer and the proof simple. For the screw displacement on a is equivalent to successive half-turns on μ and ν (in that order); and the screw displacement on b is equivalent to half-turns on ν and λ. Consecutive half-turns on μ, ν, ν, λ give the same displacement as half-turns on μ and λ only; and these give, as equivalent, the screw movement on c.

Greater symmetry of statement is obtained if the third screw movement, reversed, is taken to follow the other two; for then the three successive movements form a cycle of three successive displacements which restore the body to its original position. The figure consists of three lines a, b, c together with their common normals λ, μ, ν; the whole forming a skew rectangular hexagon $a\nu b\lambda c\mu$. The screw move-

* Halphen, *Nouvelles Annales*, Ser. 3, Tome I, p. 296 (1882).

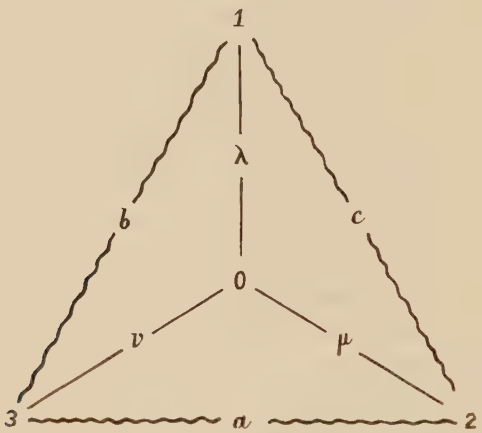
† Burnside, *Messenger of Math.*, Vol. XIX, p. 104 (1889), and Vol. XXIII, p. 19.

‡ Wiener, *Berichte*, Leipzig, Bd. 42 (1890), S. 13.

ments on a, b, c are separately and respectively equivalent to pairs of half-turns on μ and ν, ν and λ, λ and μ.

If the three positions are denoted by the numerals 1, 2, 3, the numbers may be so assigned that the screw on a connects the positions 2 and 3, the screw on b connects 3 and 1, and the screw on c connects 1 and 2. A fourth position, which may be denoted by 0, is at once associated with the three positions 1, 2, 3; being the position occupied by the body, as intermediate to any two of these, when the screw movement connecting them is replaced by two half-turns. From this parent position 0 the three positions 1, 2, 3 may be derived by half-turns on the lines λ, μ, ν respectively. This theorem is given by Halphen in extension of its equivalents, for plane and spherical displacements, previously given by Stéphanos.

The relations of the six movements and the four positions which they connect may be conveniently summarized in a diagram. Any letter lying collinear with two numerals represents the axis of the movement which connects the positions represented by the numerals. Straight lines of connection denote half-turns, and wavy lines denote screw movements.



2. *Screw Axes of the Body.*

Account has only been taken so far of the positions of the screw axes in (so called) fixed space. The better view is, of course, that the geometry is equally concerned with two rigid bodies or spaces occupying in turn three different positions relatively to each other. Any screw displacement is made about a line the position of which should be noted

in regard to both bodies. Or, if one of the bodies is to be spoken of as fixed space, then the positions of the screw axes should be noted, not only in fixed space (as has already been done), but also in regard to the moving body itself.

The screw axes of the body may be denoted by $a'b'c'$ corresponding to the screw axes abc of space. Taking the body in position 1, the passage to position 2 would be made by the screw displacement on c, with which c' coincides therefore; and the passage to position 3 would be made by the screw displacement on b, with which b' therefore coincides. Hence in position 1 the screw axes b' and c' coincide respectively with b and c. Now a half-turn about λ brings the body into position 0, without affecting the coincidence of b and b' or c and c'. Hence in the position 0 b coincides with b' and c with c'. Similarly it may be seen that in this same position a coincides with a'. Hence the three screw axes a', b', c' of the body form a figure which is congruent with the figure abc of fixed space. Completing the figure $a'b'c'$ with the addition of the common normals λ', μ', ν' a skew rectangular hexagon is formed, associated with the body, which is congruent with that of space. The two rectangular skew hexagons $a\nu b\lambda c\mu$ and $a'\nu'b'\lambda'c'\mu'$ are congruent figures, and in the zero position they coincide.

It follows therefore that any three relative positions of two bodies are obtainable by associating with them two congruent skew rectangular hexagons, placing these in coincidence, and then making half-turns about each of three alternate sides.

The identity of form of the figures formed by the screw axes of the two bodies shows that, for three positions, there is no distinction which can be drawn between the two bodies in regard to their relative positions. Either of them, relatively to the other, has the same set of positions. But for four positions (§ 12), or any greater number, the statement (which has somewhat the sound of a truism) is not true.

3. *Axodes.*

In considering the continuous motion of a body, it is common to pay regard to the surface locus traced in space and in the body by the variable screw axis of the instantaneous screw movement. The ruled surface generated by the screw axis in space is called the space-axode, and the surface generated in the body is called the body axode. The axodes touch each other along the common generator which is the axis of the instantaneous screw movement.

The notion of axodes, however, and the use of the same term, may be

usefully extended to the case of movements connecting a finite number of distinct positions. Any two positions, being connected by a screw movement, give rise to a line which is the axis of the screw; and the aggregate of these lines, in the body and in space, gives the corresponding figure of lines which is to be called an "axode." In the case of three positions it has been shown in § 2 that the axodes are congruent figures of three lines. But it is worthy of notice that, if the notion of axodes is introduced at the outset, the congruence of the axodes is almost intuitive, and the law of composition of displacements follows as an immediate corollary. This will now be shown.

4. *The Congruence of the Axodes for Three Positions.*

Let the screw axes of the first body, constituting its axode, be abc; and let the corresponding screw axes of the second body, forming its axode, be $a'b'c'$. In position 1 the passage to position 2 is made by a screw movement on the coincident lines c, c'; and the passage to position 3 is made by a screw movement on the coincident lines b, b'. The simultaneous coincidence of b with b' and c with c' shows the pair of lines b and c to be congruent with the pair b', c'. Similarly it may be shown that the pair c and a are congruent with the pair c' and a'; and that the pair a and b are congruent with a' and b'. The only way in which these three congruences may *always* be fulfilled is by the congruence of the figure abc with the figure $a'b'c'$. (Another alternative, not always possible, is considered in § 6.) Taking the congruent axodes and placing them in coincidence, the lines a, b, c coincide respectively with a', b', c'. Now, in position 1, b coincides with b' and c with c', but a does not coincide with a'. Two positions are possible in which b and c coincide with b' and c', namely, the position of complete coincidence of the axodes already taken, and the position derived from this by a half-turn about the common normal of bb' and cc'. The latter position is position 1. Positions 2 and 3 are derived similarly from the position of complete coincidence by half-turns about the common normals of cc' with aa' and aa' with bb'. Hence at once follow all the results previously given in regard to the composition of displacements and the association of a parent position from which, by half-turns, any three positions are derivable.

5. *Nuil Sets of Half-Turns and the Associated Axodes.*

A series of consecutive half-turns may restore a body to its original position. In view of what follows presently (§ 7) it is convenient at this

stage to consider the following simple cases :—

(i) *Two Consecutive Half-Turns.*—Each must reverse the other. The lines on which they take place must coincide. The axode for the body is two coincident lines, and similarly for the space-axode.

(ii) *Three Consecutive Half-Turns.*—Let the lines of the space-axode be a, b, c. If a and b did not intersect, the resultant of the two half-turns about them would be a screw displacement (or a translational displacement in the case of parallelism). This could not be neutralized by the third half-turn. Hence a and b intersect. The resultant displacement is a turn about the common normal of a and b, at their point of intersection, through an angle which is double of the inclination of a and b. If a half-turn about c neutralizes this it follows that a and b must be at right angles, and c the common normal of a and b. The three lines are three rectangular axes. The body axode, similarly, is three rectangular axes. The axodes coincide, geometrically, for each of the three positions, each half-turn reversing the sense of two of the axes. The zero position associated with the three positions also gives coincidence of the axodes. The figures and the movements are a degenerate case of the general theorem of §§ 1–3.

(iii) *Four Consecutive Half-Turns.*—Let them take place about the lines a, b, c, d, forming the space axode. The resultant of half-turns on a and b is a screw displacement on the common normal of a and b. The screw displacement given by half-turns on c and d must reverse the first. Hence the four lines a, b, c, d must have a common normal. Also, in order that the screws may give the same shift and the same turn, the mutual inclination and distance of a and b must be the opposite, respectively, of those of c and d. The figure formed by the four lines a, b, c, d and their common normal λ may be called *reversible*. Midway between a and c and equally inclined to them is a line x, normal to λ ; and a line y perpendicular to x with the same relation to a and c. The lines x and y are also, each of them, equally distant from and equally inclined to the lines a and d. The lines x, y, λ are a set of rectangular axes. The title of reversibility is justified from the fact that a half-turn about either x or y leaves the lines x, y, λ unaltered in position and exchanges the positions of a and c and also of b and d.

The body axode, similarly, consists of a reversible figure of four lines a', b', c', d' with their common normal λ'. A comparison of the two figures shows them to be congruent, and hence, in virtue of the reversibility, superposable in two ways (not counting positions as distinct if they are connected by a half-turn about $\lambda\lambda'$). In one of the superposi-

tions a, b, c, d coincide respectively with d', c', b', a', and in the other superposition a, b, c, d coincide respectively with b', a', d, c'. If a and c are the inner pair and b and d the outer pair of the first figure, then the reverse is true of the other; and conversely. These results are apparent on comparing the axodes in the four positions which they consecutively occupy in the course of the execution of the four half-turns.

6. *Two Triads of Lines Congruent in Pairs, but not in Aggregate.*

The consideration of this possibility arose in § 4, and was deferred. The question is whether two triads of lines abc and $a'b'c'$ may be congruent for every two pairs, ab with $a'b'$, bc with $b'c'$, ca with $c'a'$, without the figures being congruent in aggregate. In answering this question it will be convenient to find, more generally, what triads $a'b'c'$ exist, such that all three pairs of lines have the same distances apart and the same angles of inclination as the corresponding pairs of lines of the triad abc. Corresponding pairs of lines of the two triads may then be either congruent, or, as an alternative, symmetric (right-handed and left-handed counterparts). It will be convenient to speak of two pairs as *twins* in the one case and *images* in the other.

Let the common normals of the lines a, b, c be drawn, namely λ, μ, ν. The lengths of the sides of the skew rectangular hexagon may be denoted by the same symbols $a, \nu, b, \lambda, c, \mu$. If radii of a sphere are drawn parallel to these, and in the sense of circulation, there arises a spherical triangle representing the directions of a, b, c, with sides α, β, γ representing the inclinations of a, b, c, and the polar triangle representing the directions of λ, μ, ν. The six positive vectors $a, b, c, \lambda, \mu, \nu$, with direction and sense given by the vertices of the two spherical triangles, have a zero sum; and, conversely, to any such set of vectors with a zero sum corresponds a skew rectangular hexagon with its sides parallel and equal to the vectors. Suppose now that one of the vectors, say λ, is reversed in sign; μ and ν being unaltered, and the vectors a, b, c taking new (algebraic) values a', b', c', their resultant giving a zero sum with $-\lambda, \mu, \nu$. The new hexagon which corresponds has all its lines unaltered in direction; but, while the pairs $a'b'$ and $a'c'$ are twins of ab and ac, the pair $b'c'$ is now the image of bc.

By reversal of the sign of μ or ν two other triads are obtained, each having two pairs as twins of the original pairs, and one pair as image of the original. If two signs are reversed the new triad of lines consists of two images and one twin, as compared with the original. And, if all three signs are reversed, the new triad has all its pairs images of the original

pairs. In this last case the figure of the triad is itself the image of the original triad. (The complete set of eight triads consists of four couples of triads, each couple being images of each other.)

Suppose now that the angles α, β, γ are such as to allow a triangle to be constructed with sides respectively supplementary to α, β, γ. Then with the directions given by the vertices and those of the polar triangle, and with the same set of vectors $\pm\lambda$, $\pm\mu$, $\pm\nu$ as before, a set of eight additional triads may be constructed. Any one of these triads has each pair of lines either a twin or an image of the original pair; and, of the eight triads, all different, one must have its sides all twins of the original pairs. There arises thus a triad which, though not congruent with the original triad (for the mutual inclinations of λ, μ, ν are altered, and the lengths a', b', c' are different from a, b, c), has its three pairs of lines congruent with the pairs of lines of the original triad.

7. *The Semi-Congruent Triads as Axodes.*

Two completely congruent triads of lines were seen (§ 2) to be fundamentally related to any set of three positions, together with the parent position belonging to the three. It proves instructive to enquire into the set of positions obtainable by treating two semi-congruent triads, such as were obtained in § 6, as axodes of relative movement.

The lines b, c being congruent with b', c' may be superposed upon upon them; λ being consequently superposed on λ'. There are two positions obtainable in this way, the two being connected by a half-turn on $\lambda\lambda'$. The positions may be called 1 and 1′. Superposition of lines c, μ, a upon c', μ', a' give positions 2 and 2′; and superposition of a, ν, b upon a', ν', b' give positions 3 and 3′. These six positions may be presented as a cycle of six, in the order 2′, 3, 1′, 2, 3′, 1, the connecting movements taking place on screws with axes a, b, c, a, b, c in the one figure, and a', b', c', a', b', c' in the other. For consider position 2′, in which c and a coincide with c' and a', b and b' being non-coincident. As the pair ab are congruent with the pair $a'b'$, a screw movement on aa' will bring b to coincide with b', separating at the same time c and c'. The new position is either 3 or 3′, and for convenience of notation it may be taken to be 3. The next position 1′ is reached by a screw motion on bb', and so on. The complete cycle of six positions is in this way obtained after using two complete circuits of the screw axes a, b, c. [A conceivable exception would occur if the fourth position in the sequence were identical with the first, breaking the cycle into two separate cycles of three. But this result would be in conflict with the fact already established (§ 4) that the move-

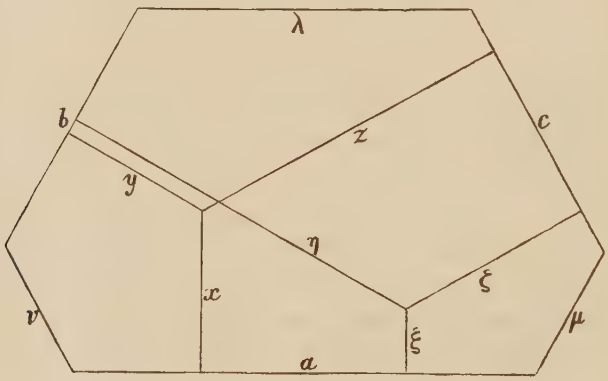
ments connecting three positions are accounted for by axodes which are completely congruent.]

8. *Rectangular Axes Associated with the Semi-Congruent Axodes.*

Consider the transition from position 2 to position 3, with position 3′ as a third position associated with them. The movement connecting 3′ with each of the others is already known. The movement connecting 2 and 3′ is a screw movement on aa'; and the movement connecting 3′ and 3 is a half-turn on $\nu\nu'$. Now $\nu\nu'$ is normal to aa'. Hence the screw movement may be replaced by half-turns on two normals to a, of which ν is one and x, say, is the other. The movement connecting positions 2 and 3 is therefore (§ 1) a half-turn on x. Similarly the positions 3 and 1 are connected by a half-turn about a line y normal to b; and positions 1 and 2 are connected by a half-turn about a line z normal to c. (When each line comes into operation the corresponding accented line of the second figure coincides with it.) Thus half-turns about the three lines x, y, z connect the positions 1, 2, 3. It follows (§ 5) that the lines x, y, z are concurrent and mutually perpendicular. Similarly, in the second figure, the lines x', y', z', normal respectively to a', b', c', are concurrent and mutually perpendicular.

Taking the positions 1′, 2′, 3′ in the same way, it appears that a second set of rectangular axes ξ, η, ζ connect these positions by half-turns, and that ξ, η, ζ are respectively normal to a, b, c.

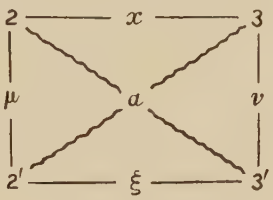
The complete figure, which is shown schematically only, might be described and constructed in reverse order: x, y, z and ξ, η, ζ being taken arbitrarily as two sets of rectangular axes; the lines a, b, c being the common normals of the pairs x and ξ, y and η, z and ζ; and λ, μ, ν being the common normals of b and c, c and a, a and b. Similarly also for the second figure.



9. *Axial Symmetries of the Axode Figure.*

Take in sequence the four positions 2, 2′, 3′, 3. These are connected by movements all of which are half-turns. The half-turns take place, in order, about the lines μ, ξ, ν, x. It follows (§ 5) that the four lines have a common normal (which is the line a), and that the two pairs of lines $\mu\nu$ and $x\xi$ form a reversible figure. The same is true of the pairs $\nu\lambda$ and $y\eta$, normals of b, and also of the pairs $\lambda\mu$ and $z\zeta$ normals of c. This three-fold property of axial symmetry may be stated of the figure constructed as described at the end of § 8, without any reference to the displacements associated with it.

The six movements connecting the four positions dealt with in this present paragraph may be represented in a diagram after the fashion of that in § 1.



And if the six triads of numerals and letters are circulated simultaneously two other such diagrams are produced. The three together exhibit completely all the fifteen displacements connecting the six positions. Each screw displacement may be expressed as the resultant of two of the half-turns. The screw on a which converts 2 to 3′ is equivalent to half-turns (in order) on μ and ξ, and also to half-turns on x and ν. These are equivalent in virtue of the reversibility of the pairs $\mu\nu$ and $x\xi$. The screw on a which converts 2′ to 3 is equivalent to half-turns on μ and x, and also to half-turns on ξ and ν; and these are equivalent in virtue of the same reversibility.

10. *Dissection of the Axodes into Congruent Skew Pentagons.*

Comparison of the two axodes in position 1 shows that the skew rectangular pentagon $yzc\lambda b$ is superposed on the corresponding pentagon of the second figure. The normal x at the vertex yz coincides, at the same time, with the corresponding line x' normal at $y'z'$. A similar result is obtained from each of the other positions. The six pentagons, associated one with each position, are as follows :—

Position 1 : pentagon $yzc\lambda b$.	Position 1′ : pentagon $\eta\zeta c\lambda b$.
Position 2 : pentagon $zxa\mu c$.	Position 2′ : pentagon $\zeta\xi a\mu c$.
Position 3 : pentagon $xyb\nu a$.	Position 3′ : pentagon $\xi\eta b\nu a$.

The six pentagons may be denoted (1), (2), (3), (1′), (2′), (3′). Placed edge to edge they constitute material out of which the whole figure may be built up. The second figure is also built up, though differently, out of the same material. It is of interest to examine the difference of construction which produces the non-congruent axodes.

Take position 1. In this position the passage to 2 is made by a half-turn about z. In position 2 the pentagon (2) coincides with the corresponding pentagon of the second axode. Hence if, dislocating the first axode, the pentagon (2) is given a half-turn about z, it then occupies the same position relatively to (1) as occurs in the second axode. Similarly, if (3) is given a half-turn about y it has then the same position relatively to (1) as occurs in the second axode. The change of relative position of (2) and (3) produced by these two changes is equivalent to a half-turn about x. Hence it follows that if the first axode is dissected, then in order to put together the three pieces (1), (2), (3) afresh so as to match the second axode, the same hinge-lines x, y, z are used, but the right angles between the planes of these pairs of hinges are reversed. (Three right angle pieces, with a given correspondence of hinges along the edges, can be put together in two ways.) The same is true of the three pentagons (1′), (2′), (3′), (3) forming the other half of the axode; they are to be put together with the same hinge lines ξ, η, ζ, but reversely. The two halves of the new figure thus made may then be put together, fitting along the lines of the hexagon $a\mu c\lambda b\nu$. In this last stage of the new construction it will happen of itself that, as regards the pair of pieces (1) and (1′), the relative position which they occupied in the first axode will have suffered a half-turn; and similarly for the other pairs (2) and (2′), (3) and (3′). The two axodes are built up of six identically congruent pieces, each a skew pentagon, but along each of the hinge lines x, y, z ξ, η, ζ, λ, μ, ν, the fit is reversed by a half-turn. The operation of converting the one figure into the other might be described as "turning it inside out."

The possibility of converting such a figure as is described at the end of § 8 into a fresh figure by nine reversals of fit is itself a geometrical theorem apart from the system of displacements associated with the figure.

11. *Spherical Figures.*

If spherical figures are taken representing the directions of the lines of the geometry of §§ 6–10, and disregarding their distribution in space, there appear the relative movements of two spherical triangles with

supplementary sides. They take up in succession six positions by rotations about vertices abc and $a'b'c'$, corresponding pairs of which are, for each rotation, either coincident or antipodal. The half-turns into which the movements analyse take place about the vertices of the polar triangles $\lambda\mu\nu$ and $\lambda'\mu'\nu'$, and about the vertices of quadrantal triangles xyz and $\xi\eta\zeta$ inscribed in the triangle $\lambda\mu\nu$ (when coinciding with the corresponding points of the second figure). The three-fold reversibility occurring in the solid figure (§ 9) shows here that $\mu\nu$ and $x\xi$ have, as pairs of directions, common angle bisectors; and similarly for $\nu\lambda$ and $y\eta$, $\lambda\mu$ and $z\zeta$. Hence the circumcentre of the triangle $\lambda\mu\nu$ is the centre of rotation which connects the positions of the quadrantal triangles xyz and $\xi\eta\zeta$. The theorem of spherical geometry which results is this :—A quadrantal triangle xyz is rotated into a new position $\xi\eta\zeta$. Then the triangle formed by the arcs of great circles $x\xi$, $y\eta$, ζz has as its circumcentre the centre of rotation.

12. *Axodes for any Four Positions.*

Special cases of a set of four positions have already occurred. The general case may be briefly considered. The four positions will be denoted by the numerals 1, 2, 3, 4. The passage between the positions 1 and 2 is made by a screw movement on a line which, as a line of the first body, will be called 12. The axode of the body consists of the six lines 12, 13, 14, 23, 24, 34. The common normal of 12 and 13 will be denoted by (14), the common normal of 24 and 34 by (41). This notation accounts for twelve of the common normals, and the remaining three are not essential. With any three of the four positions is associated a parent position (§§ 2–4). The parent position of 1, 2, 3 will be denoted by [4], and similarly for the others. From position [4] the positions 1, 2, 3 are obtained by half turns on (14), (24), (34) respectively, and similarly for the other positions. Position 4 is obtainable from positions [1], [2], [3] by half-turns on (41), (42), (43) respectively. The relation between the positions 1, 2, 3, 4, and [1], [2], [3], [4] is mutual; each set are the parent positions derived from the other set.

In position 1 the lines 12, 13, 14 coincide with the corresponding lines of the second axode, the two triads being therefore congruent. And in position [1] the lines 23, 34, 42 coincide with the corresponding lines of the other body, and are therefore congruent with them. Thus, dividing the six lines of the axode into the two triads 12, 13, 14 and 34, 42, 23, the corresponding triads of the second axode are respectively congruent with them. This congruence of the two axodes, when broken into two

triads of lines, occurs in four ways. The axodes are not, however, congruent in aggregate.

A property of symmetry occurring in either axode may also be noticed. Let the positions 1, [3], 2, [4] be taken as a cycle of four. The movements connecting them are all half-turns, the sequence of lines on which they take place being (13), (23), (24), (14). These have the common normal 12, and (§ 5) the two pairs of lines (13), (24) and (23), (14), with their normal 12, must form a reversible system. The two pairs of lines are respectively the common normals of 12 with the pair 23, 14 and the pair 13, 24. The axial symmetry expressed by the property of reversibility holds good in turn for each of the six lines and the four normals which meet it. The same is true for the second axode.

13. *Centrodes for Four Positions.*

The theorems of the last paragraph take degenerate forms when the displacements are spherical or plane displacements. The results are very similar in the two cases, and the latter alone is now referred to. Each centrode figure consists of six points, centres of rotation for the passage between the four positions. The triangles formed by the two triads of points 12, 13, 14 and 34, 42, 23 are equal to the corresponding triangles of the second centrode figure, being directly equal in the former case and reversely in the latter. Such pairs of equal triangles are obtainable in each of four ways. Further, the lines joining the point 12 to the pair of points 13, 24, and the lines joining 12 to the pair of points 23, 14 have the same angle bisectors. A similar result holds good for each of the six points of the centrode figure. The same is true of the second figure.

14. *The General Projective Equivalent of the Plane Centrode Figure.*

A figure with the properties just described may be readily derived, it may be of interest to note, from the space figure of two mutually inscribed tetrahedra. Taking $ABCD$ and $A'B'C'D'$ to be the vertices of the two tetrahedra, consider any plane section of the figure. Any one of the planes, say $ABCD'$, cuts the plane of section in a straight line on which appears an involution of three pairs of points; those, namely, in which the sides of the quadrangle $ABCD'$ meet the line. The plane figure made by the plane of section is therefore a figure given by four pairs of lines having the property that any three pairs of lines cut either of the other two lines in involution. In drawing such a figure only three of the involu-

tions are used in making the construction, and the remaining five are properties of the figure so drawn. If, say, the four pairs of lines are $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ and $\delta\delta'$, and if α is cut in involution by the three pairs $\beta\beta'$, $\gamma\gamma'$, $\delta\delta'$, and similarly for β and γ, then the same is true also for δ, α', β', γ', and δ'.

The correlative theorem, in terms of points, is this:—Four pairs of points AA', BB', CC', DD' in a plane are such that the pairs BB', CC', DD' projected from A give a pencil in involution, and similarly for B and C. It follows then that the pairs BB', CC', DD', projected from A', give a pencil in involution, and similarly for B', C', D' and D.

If in this last figure one pair of points, say D and D', are made the circular points at infinity, then the bisectors of the angles subtended at A by BB' and CC' are the same, and similarly for every other point. It is (§ 13) the centrode figure for the displacements connecting four positions of a two-dimensional body.

ON SEMI-INTEGRALS AND OSCILLATING SUCCESSIONS OF FUNCTIONS

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§§ 1–4. *Introductory.*

1. The introduction of the concept of upper and lower semi-continuity by Baire was a happy one. It has enabled us, not only to deal with continuity itself with much greater ease, but it has rendered possible the discovery and statement in concise form of theorems which might otherwise not have been suspected. One branch of the Theory of Functions of a Real Variable in which it is particularly useful is in the theory of oscillating series. To mention only one of the more recent results obtained by myself in this domain, we have the theorem that what may be called* semi-integration below of an oscillating succession is allowable if (i) the points in the neighbourhood of which the non-uniformity of the lower oscillation of the succession is infinite form a countable set, and (ii) the lower function of the integrals of the succession is upper semi-continuous on the left and lower semi-continuous on the right.

We owe to Jordan the recognition of the importance of a certain class of continuous functions, namely, the class consisting of those continuous functions which possess bounded variation (*fonction à variation bornée*). All continuous functions obtained by Lebesgue integration belong to a sub-class of this class. In fact, a function which is an integral, interpreting the word "integral" in the most general Lebesgue sense, is not only a function with bounded variation, but the increment over any set of intervals has zero as unique limit, when the content of the set of intervals diminishes indefinitely.

The recognition of this fact has enabled Vitali to obtain the following result.† If the points at which a sequence of functions converges non-

* If $l(x)$ is the lower function of the succession, and $L(x)$ of the succession of integrals, semi-integration below implies that $\int l(x)\,dx \leqslant L(x)$.

† Stated a little differently by Vitali.

uniformly with infinite measure form a set of zero content, then term-by-term integration is allowable, provided the integrals of the functions converge towards a function which is an integral. If it were not an integral term-by-term integration would lead to a result absurd on the face of it; thus we might say that under the circumstances stated with regard to the non-uniformity of the convergence, term-by-term integration is allowable, if it has any sense. Striking as this result is, we may say *en passant* that the determination of whether or not a function is an integral, especially when, as in the present case, it is defined as a limit of succession of functions, is so much more difficult than the determination of whether or not a function is continuous, that Osgood's theorem as to the term-by-term integration of sequences, of which the generalisation for oscillating successions has been stated above, remains unimpaired in importance, as well as in interest. Osgood's theorem may, indeed, be also thought of as giving a sufficient condition under which a sequence of functions each of which is an integral, has another integral as limiting function, viz., this is the case if (i) the limiting function is continuous, and (ii) the points of infinite non-uniform convergence are countable. The question then arises of itself, what is the generalisation for successions of functions of Osgood's theorem, stated in this last manner, more especially when we consider semi-integration of the succession above or below only? We naturally expect to be able to treat the concept of being an integral in the same way as that of continuity, viz., to be able to split it up into two constituent parts. When we have done this we may hope to be able to obtain the generalisation for oscillating successions of Vitali's theorem.

With this object in view I introduce in the present paper the concepts of upper semi-integral and lower semi-integral. This use of terms seems appropriate in view of the fact that it follows at once from the definition that an upper integral of a function (in Darboux's sense) is an upper semi-integral of the integrand, though it is not in general associated with it, using the word "associated" in a sense to be explained later on (§ 12). There is thus no confusion. A function which is both an upper semi-integral and a lower semi-integral is certainly an integral, and it will be found that the various properties of an integral are distributed between the classes of upper and lower semi-integrals.

The simplest definition which can be given of these new concepts is that an upper semi-integral is the sum of an integral and a positive monotone increasing function,* while a lower semi-integral is the

* "Positive" here means $\geqslant 0$, and "monotone decreasing" means monotone and not increasing. Similarly "negative" means $\leqslant 0$, and "monotone increasing" means monotone and not decreasing.

sum of an integral and a negative monotone decreasing function. Thus, as we naturally exclude infinite values and consider closed intervals, semi-integrals are, like integrals, functions with bounded variation.

On the other hand, it is important to remark that they are not necessarily continuous. We find it convenient, in fact, to simultaneously split up the concepts of integration and continuity; the latter concept is, however, split up in the special manner which the theorem concerning oscillating sequences, more than once referred to, would lead us to expect, viz., it follows from the definition that an upper semi-integral is upper semi-continuous on the left and lower semi-continuous on the right, while a lower semi-integral is lower semi-continuous on the left and upper semi-continuous on the right.

The disadvantage of the definition of semi-integrals just given is that it requires the previous definition of the term integral. Jordan's separation of the positive and negative variations, however, enables us immediately to give independent definitions, as soon as we adapt to these separate variations the statement of the property of an integral relative to the total variation. Thus *we define an upper (lower) semi-integral to be a function with bounded variation, whose negative (positive) increment over any set of intervals has zero as unique limit, when the content of the intervals decreases indefinitely.*

2. The introduction of these concepts enables us to split up into two constituent parts Lebesgue's "Fundamental Theorem of Integration." Moreover, in this way it becomes possible to give an independent account of the chief properties of semi-integrals. Among these we may mention the property that an upper semi-integral exceeds, and a lower semi-integral is less than, the common integral of any of its derivates by a monotone increasing function. If the sum of two semi-integrals of the same kind is an integral, each of the semi-integrals is an integral.

The limit of an ultra-monotone* sequence of upper (lower) semi-integrals is an upper (lower) semi-integral, whether the sequence be ascending or descending, provided only the sequence of integrands does not diverge, except possibly at a set of points of content zero. If a function of bounded variation has one of its derivates finite above (below)†

* An ultra-monotone sequence is defined to be a monotone sequence such that the difference between any two consecutive terms is a monotone function, the sense of the monotony being preserved.

† "Finite above" implies that the function is never $+\infty$, "finite below" that it is never $-\infty$.

it is a lower (upper) semi-integral, and the integral of that finite derivate over the set of points where it is positive (negative) is the positive (negative) variation of the function.*

3. These properties themselves, though easily proved by anyone who has mastered Lebesgue's work, seem to be sufficiently striking to appear to justify the introduction of the new terms; but their *raison d'être* appears more especially, as already explained, when we come to the theory of integration of oscillating successions. We have, for example, the theorem that, if the limiting functions of the integrals of a finitely oscillating succession of functions are greater (less) than or equal to the integral of the lower (upper) functions of the succession, or, with our phraseology, if semi-integration of the succession is legitimate below (above), then all these limits are necessarily upper (lower) semi-integrals, from which it of course follows that, if semi-integration is allowable, both above and below, the limits are integrals. Moreover, we have the theorem that the necessary and sufficient condition that a succession of functions whose points of infinite non-uniform oscillation below (above) form a set of zero content, should be semi-integrable below (above) is that the lower (upper) function of the succession of integrals should be an upper (lower) semi-integral. That under the same conditions semi-integration should be possible both above and below, it is necessary and sufficient that the upper and lower functions of the succession of integrals should both be integrals.

We have, moreover, the answer to the question as to the generalisation of the second portion of Osgood's result, viz., if the succession of integrands oscillates finitely, and the points at which it has non-uniform oscillation with infinite measure below (above) are countable, then the necessary and sufficient condition that the upper and lower limits of the integrals should be both upper (lower) semi-integrals, is that they should be upper (lower) semi-continuous on the right and lower (upper) semi-continuous on the left. Under the same conditions the condition that they should be integrals is that they should be continuous.

It is here shewn that the necessary and sufficient condition that a succession of functions, oscillating finitely, should be semi-integrable

* For a proof of this theorem in the modified form in question the reader may refer to "Functions of Bounded Variation," a forthcoming paper in the *Quarterly Journal*, by the author, in which, moreover, beside certain new results, new proofs (the first in English) will be found of theorems due to Lebesgue, and required in the present paper. These proofs differ from Lebesgue's, among other respects, in that, with one exception, they do not involve the use of Cantor's numbers.

above (below) is that given any positive e, we can find E, such that the positive (negative) increment of each of the integrals of the succession of integrals taken over any set of intervals whose content is less than E should be less than e. Moreover we have a generalisation of the other result of Vitali's in this connexion, viz., that, if the succession is bounded below (above), the condition in question is also necessary and sufficient for the semi-integrability (not absolute) of the succession term-by-term.

It will be seen that, though several of the generalisations with which this paper is concerned are almost intuitive to anyone accustomed to deal with oscillating successions, as soon as the preliminary step involved in the splitting up of the notion of integral has been taken, this is by no means the case with all of them. They are much more intuitive, apparently, than the theorems of my paper on "Term-by-Term Integration of Oscillating Series."* Yet, when the step in question has been taken, it is found that some of the theorems true for sequences do not admit of generalisation. This fact and the great importance of the subject will, I hope, be thought to have justified the detailed treatment here given to the matter.

4. Among the results of this paper to which attention may be directed we have the following remarkably convenient necessary and sufficient test for the term-by-term integrability of sequences :—a sequence is integrable term-by-term if two other integrable sequences can be found, one of which is greater than the given sequence, that is, such that the constituent function is greater than or equal to that of the given sequence, and the other sequence is less than the given sequence. This theorem will, I feel sure, be found in practice of great utility. It is noteworthy that it is not sufficient to find two sequences, one greater than the given sequence and integrable above, and the other less than the given sequence and integrable below, unless these two sequences are bounded respectively above and below. Moreover the corresponding tests for successions do not apply as they stand. The nature of the difficulty will be apparent, after reading the paper; it depends on the fact, here proved by an example, that if a succession of essentially positive functions is semi-integrable above, it is not necessarily absolutely semi-integrable above, whereas an integrable sequence of essentially positive function is necessarily absolutely integrable.

It is noteworthy that the test in question holds good, whether the range of integration be a finite or an infinite interval, and, though in

* *Proc. London Math. Soc.*, Ser. 2, Vol. 8, pp. 99–117.

practice this is less important, whether the integrals are absolutely convergent, or not. Thus it enables us not only to determine with ease in the great variety of cases whether or not an infinite series which is capable of term-by-term integration over a finite interval can also be integrated when one or both of the limits of integration are infinite; it enables us also to discuss questions of primary importance in the theory and practice of definite integrals between infinite limits and involving parameters.

Want of space alone has prevented me enlarging in the present paper on this point. I have, however, illustrated in more ways than one the use of the test, *inter alios* by obtaining new results with regard to the continuity of a parametric integral, and the possibility of differentiating under the sign of integration.

On Integrals and Semi-Integrals.

5. We use the term "integral" in the most general sense consistent with the definition of Lebesgue,* and assume that all functions which occur, bounded or unbounded, possess integrals.† One of the advantages of this generalisation, whether we adopt Lebesgue's definition, or one given by myself,‡ is that we are naturally led to consider the behaviour of the function at the points of certain sets, not necessarily filling up an interval, or a set of intervals. In this connection it is convenient if we define with him the integral of a function over a set of points contained in an interval as the integral over that interval of the function got from the original function by ascribing the value zero to it at the points of the set complementary to the set in question, and leaving it unaltered elsewhere. With this understanding we have the following theorem, which is needed in the sequel :—

THEOREM.—*If $f(x)$ is a summable function, its integral over any measurable set of points E exists, and, if D_1, D_2, ... are sets of intervals, each lying inside the preceding, and having the points of E as common points, and no other common points, except perhaps a set of content zero, then the integral of $f(x)$ over the set E is the unique limit of its integral over the sets D_1, D_2, ...*

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* H. Lebesgue, "Intégrale, Longueur, Aire," 1902, *Ann. di Mat.*

† Such functions are called by Lebesgue "summable functions," *Leçons sur l'intégration*, 1904.

‡ W. H. Young, "The General Theory of Integration," 1904, *Phil. Trans.*, Series A, Vol. 204, pp. 221–252. "A New Method in the Theory of Integration," 1910, *Proc. London Math. Soc.*, Series 2, Vol. 9, pp. 15–50.

6. Turning to the consideration of the integral itself, we remark that a function which is an integral has various well-known and striking properties which distinguish it from other continuous functions. Denoting the function which is equal to the integrand wherever positive (negative) and is elsewhere zero, as the positive (negative) part of the integrand, the integral is from the very definition the difference between the integrals of two essentially positive functions, viz., the positive part and the absolute value of the negative part of the integrand, and is therefore the difference between two monotone increasing positive continuous functions, or, which is the same thing, the sum of a monotone increasing and a monotone decreasing continuous function. Thus it also follows that an integral has bounded variation. Though, however, the integral of a summable function is necessarily continuous and possesses these properties, a continuous function possessing these properties is not, as has been pointed out by Lebesgue and Vitali, necessarily an integral. Lebesgue gives the following necessary and sufficient conditions that a function should be an integral :—

(*a*) *The function must have bounded variation.*

(*b*) *Its total increment over any set of non-overlapping intervals of content I, must have zero as its limit as I approaches zero.**

It will be seen from what follows that the following is an equivalent necessary and sufficient test, which, though logically inferior, may be in practice much more convenient :—

It is necessary and sufficient that (1) *a function should exist which is an integral and exceeds the given function by a monotone increasing function, and* (2) *a function should exist which is an integral and which is less than the given function by a monotone increasing function.*

It may be remarked that in this form the test is complete ; if it is satisfied, the function is necessarily continuous and of bounded variation.

We note at once, moreover, how the very form of this test suggests the possibility of splitting up the property of being an integral into two parts. We shall say, in fact, that a function is a lower semi-integral if it satisfies the first part of the test, and an upper semi-integral if it satisfies the second.

We are thus naturally led to split up Lebesgue's test. Before doing

* H. Lebesgue, *Leçons sur l'intégration*, p. 129, footnote (statement only) ; "Sur la recherche des fonctions primitives par l'intégration," *Rend. dei Lincei*, Vol. xvi, 1907, pp. 286, 287 ; "Sur les Intégrales singulières," 1909, *Ann. de Toulouse*, 3rd Series, Vol. I, p. 43. G. Vitali, "Sulle funzioni integrali," *Atti d. Torino*, Vol. xl, 1905, pp. 1021–1034.

so, however, it will be convenient to give a proof of the *necessity* of Vitali's condition, which includes Lebesgue's condition (*b*) and entails condition (*a*) as a consequence,* viz.,

THEOREM.—*A function $F(x)$ which is an integral is absolutely continuous, in other words it is such that, if we take any finite or countably infinite set of non-overlapping intervals of content less than a suitably chosen quantity E, the positive and negative increments taken over this set of intervals are each numerically less than a pre-assigned arbitrary positive quantity e.*

When the integrand is bounded the theorem is obvious. To prove it in the general case we remark that, if a function exceeds another by a monotone increasing function, the positive increment of the former over any set of non-overlapping intervals is necessarily greater than or equal to that of the latter.

Denoting then, by f_1 and f_2, the positive and negative parts of f, that is, the two positive functions agreeing numerically with f wherever $f \geqslant 0$ and $f \leqslant 0$ respectively, and zero elsewhere, these are both summable functions, and

$$f(x) = f_1(x) - f_2(x),$$

and, denoting the integrals by large letters,

$$F(x) = F_1(x) - F_2(x);$$

hence, by what was pointed out above, the positive increment of $F(x)$ over any set of intervals is less than or equal to the corresponding positive increment of $F_1(x)$. Thus the theorem for the positive increment of $F(x)$ is true, if it is true for $F_1(x)$. It is therefore only necessary to prove the theorem in the case when the integrand is positive.

Let $a_n(x) = f(x)$ wherever the positive function $f(x)$ is less than n, and elsewhere let $a_n(x) = 0$. Then

$$a_1(x) \leqslant a_2(x) \leqslant \ldots \leqslant a_n(x) \leqslant \ldots$$

is a monotone increasing sequence, having $f(x)$ as limit, and the integral of f is (by definition if this integral is an improper integral) the limit of the integral of $a_n(x)$ over any interval.

Hence pre-assigning a positive quantity e, we can find an integer n, such that

$$\int a_n(x)\,dx \leqslant \int f(x)\,dx \leqslant \tfrac{1}{2}e + \int a_n(x)\,dx;$$

* See "Functions of Bounded Variation," *loc. cit.*

and therefore $$0 \leqslant \int \{f(x)-a_n(x)\}\,dx \leqslant \tfrac{1}{2}e.$$

Since this is true over the whole interval of integration it is true over every set of intervals D in the main interval, since the integrand is positive. Thus

$$0 \leqslant \int_D f(x)\,dx \leqslant \tfrac{1}{2}e+\int_D a_n(x)\,dx.$$

Now $a_n(x)$ lies between 0 and n, therefore this gives

$$0 \leqslant \int_D f(x)\,dx \leqslant \tfrac{1}{2}e+nD.$$

Thus, if the set of intervals was such that $D < \tfrac{1}{2}e/n$, this gives

$$0 \leqslant \int_D f(x)\,dx < e,$$

which proves the required result for the positive increment; similarly it may be proved for the negative increment.

7. The property just proved is not a property of every continuous function which is the sum of a monotone increasing and a monotone decreasing function. We notice, however, that every monotone increasing function, continuous or not, possesses one half of the property, since the negative increment is zero, while every monotone decreasing function possesses the other half. Moreover, it is evident that a function which is the sum of two functions both of which possesses the same half-property, will itself possess it. Bearing in mind certain facts to be proved shortly, we are thus led to formulate the following definition, which for the rest has already suggested itself.

DEFINITION.—*A function which can be expressed as the sum of an integral and a monotone increasing function (continuous or not) is called an upper semi-integral, and one which can be expressed as the sum of an integral and a monotone decreasing function (continuous or not) is called a lower semi-integral.*

8. With this definition, a theorem which I have proved elsewhere* shews that, in particular, *a continuous function of bounded variation, having a derivate which is finite above (below) is an upper (lower) semi-integral, and the integral of the positive (negative) part of that derivate is the positive (negative) variation of the semi-integral over the segment considered.*

Again, *a semi-integral possesses a finite differential coefficient*, except at a set of content zero, since, by a theorem of Lebesgue's* this is a pro-

* See "Functions of Bounded Variation," *loc. cit.*

perty of any function of bounded variation, and *the derivates of a semi-integral are all summable functions, whose common integral has, therefore, by another theorem of Lebesgue's* the same differential coefficient as the semi-integral except at a set of content zero.*

Hence, if $F(x)$ is an upper semi-integral, and $g(x)$ any one of its derivates, and $I(x)$ and $M(x)$ the integral and monotone increasing function whose sum is $F(x)$, we have, except at a set of content zero, differentiating

$$g(x) = I'(x) + M'(x),$$

whence, integrating,

$$\begin{aligned}\int_a^x g(x)\,dx &= \int_a^x I'(x)\,dx + \int_a^x M'(x)\,dx \\ &= I(x) + M(x) - \text{a monotone increasing function,}^* \\ &= F(x) - \text{a monotone increasing function;}\end{aligned}$$

from which we see, by what has already been pointed out, that this last monotone increasing function has a differential coefficient which is zero, except at a set of content zero.

Now a finite function which is zero except at a set of content zero cannot be a derivate without being identically zero, by Lebesgue's theorem, or by the analogous theorem stated at the beginning of the present article. Hence, if $F(x)$ is not itself an integral, the differential coefficient just mentioned has infinite values.

Moreover, by a familiar extension* of the theorems used, these infinities cannot form a reducible† set.

A similar discussion may be made for a lower semi-integral. Thus we have the following theorem: –

THEOREM.—*An upper (lower) semi-integral which is not an integral is equal to the common integral of its derivates plus a monotone increasing (decreasing) function having zero as its differential coefficient, except at a set of content zero, and an infinite derivate at an irreducible set of points.*

9. THEOREM.—*An upper (lower) semi-integral has the property that, if we take any finite or infinite set of non-overlapping intervals of content less than a suitably chosen positive quantity E, the negative (positive*

* See "Functions of Bounded Variation," *loc. cit.*

† [*Note added February 18th*, 1911.—They form indeed a non-countable set. Moreover there is at least one perfect set consisting entirely of such infinities. This follows from a theorem stated and proved in my "Note on the Fundamental Theorem of Integration," presented on November 4th, 1910, to the Cambridge Philosophical Society.]

increment of the function taken over those intervals is numerically less than any pre-assigned arbitrary positive quantity e.

This statement corresponds to Vitali's form of the condition that a function should be an integral. Corresponding to Lebesgue's form we have the following alternative statement :—

Theorem.—*An upper (lower) semi-integral is a function of bounded variation, and has the property that its negative (positive) increment over any set of non-overlapping intervals of content I, must have zero as its limit as I approaches zero.*

Either of these theorems, however, expresses only a portion of the information within our reach as to the nature of semi-integrals. The converse is also true, viz., the property, or pair of properties, in question is characteristic of semi-integrals.

Since a function which satisfies the conditions of the former theorem (Vitali's form of the condition) satisfies the requirements of the alternative theorem (Lebesgue's form of the condition), it will be sufficient to prove the following theorem, in order to justify the statement just made.

Theorem.—*If $F(x)$ is a function of bounded variation, such that its negative (positive) increment over any set of non-overlapping intervals of content I has zero as its limit as I approaches zero, then $F(x)$ is an upper (lower) semi-integral.*

Since $F(x)$ has bounded variation it has a finite differential coefficient, except at a set of content zero. Let $F'(x)$ be the function which is equal to this differential coefficient except at the exceptional set, and there is zero. Then

$$\int F'(x)dx$$

has F' for differential coefficient except at a set of content zero. Therefore, if

$$G(x) = F(x) - \int F'(x)dx,$$

$G(x)$ has zero for differential coefficient, except at a set of zero content. But also the negative increment of $G(x)$ over any set of intervals of content less than E has zero as limit, when E diminishes indefinitely ; therefore, describing round the exceptional points at which the derivates of $G(x)$ are different from zero a set of non-overlapping intervals of sufficiently small content, the increment of $G(x)$ over these intervals is greater than $-e$, where e is as small a positive quantity as we please. Hence, by the ordinary reasoning in which a Lebesgue chain of intervals is employed, it follows that $G(x) - G(a)$ is essentially positive, whatever interval (a, x)

be considered. Thus $G(x)$ is a monotone increasing function of x, and therefore $F(x)$ is the sum of an integral and a monotone increasing function of x.

*Hence also it follows that a function with bounded variation, whose positive and negative increments over a set of non-overlapping intervals of content I both have zero as limit when I is indefinitely decreased, is an integral.**

We also have the alternative statement:—

A function such that, pre-assigning any positive quantity e, we can find a positive quantity E, such that the positive and negative increments of the function over any set of non-overlapping intervals of content less than E are both less than e, is an integral.†

10. We may now enunciate the second definition of upper and lower semi-integrals, and it will be evident from what precedes that it will be concomitant with the first.

DEFINITION.—*A function of bounded variation, such that its negative (positive) increment over any set of non-overlapping intervals of content I has zero as limit, when I approaches zero, is called an upper (lower) semi-integral.* Or, as an alternative, *a function such that, pre-assigning any positive quantity e, we can find a positive quantity E, such that the negative (positive) increment of the function over any set of non-overlapping intervals of content less than E is numerically less than e, is called an upper (lower) semi-integral.*

11. The mode of proof adopted in § 9 leads again to the theorem given at the end of § 8.

We have, moreover, the following theorems:—

THEOREM.—*A function which is both an upper and a lower semi-integral is an integral.*

In other words, if a function is such that it is less than an integral by a monotone increasing function, and greater than an integral by a monotone increasing function, it is itself an integral. This gives a test, which may in certain cases be extremely convenient, for ascertaining whether a function is an integral.

THEOREM.—*If the sum of two upper (lower) semi-integrals is a lower*

* Lebesgue, *loc. cit.*

† Vitali, *loc. cit.*

(upper) semi-integral, each of the semi-integrals is an integral. In particular, if the sum of two monotone increasing (decreasing) functions is an integral, each of those functions is an integral, and of course continuous.

12. We have seen that semi-integrals are characterised by the fact that they are obtained from integrals by a process of addition or subtraction of a monotone increasing function. The same semi-integrals will in this way be obtained from various integrals; there are, however, semi-integrals which may be said to belong to, or to be associated with, a particular integral. We have seen, in fact, that a semi-integral shares with only one integral, to a constant *près*, the property of having a differential coefficient which coincides with that of the integral except at a set of content zero. Thus every integral has an associated set of upper and lower semi-integrals.

Thus we may conveniently give the following definition:—

DEFINITION.—*The associated semi-integrals of an integral are called the semi-integrals of the integrand.*

Thus every summable function has integrals and semi-integrals. The upper (lower) semi-integrals are all of them greater (less) than the definite integral, but their differential coefficients all exist and are finite and summable and agree with the integrand, except at a set of content zero.

13. It will now be necessary to consider certain portions of the theory of oscillating successions of functions. We say that such a succession *oscillates finitely above (below)*, if there is no point at which the upper (lower) function is $+\infty$ $(-\infty)$. It is mostly with such successions that we shall have to deal, and if nothing is said to the contrary, it may be assumed that this is the case.

If a succession of functions is such that the limiting functions of the integrals of the succession are all less (greater) than, or equal to, the integral of the upper (lower) function of the original succession, this latter succession is said to be *semi-integrable above (below)*. The integrals in this case are supposed taken over any interval (a, x) whatever. If, and only if, the succession is semi-integrable both above and below, and further the upper and lower functions agree except at a set of content zero, all the limits of the integrals will coincide with the common integral of the upper and lower functions, in this case the succession is said to be *integrable term-by-term*, or, shortly, to be *integrable*.

14. THEOREM.—*If $f_1 \leqslant f_2 \leqslant \ldots$ is a monotone ascending sequence* of essentially positive functions, which diverges at a set of points of content greater than I, then we can assign an integer m, such that, for all values of $n \geqslant m$, the set of points at which $f_n(x) > H$ is of content greater than I.*

For, if x is a point at which $f(x) > H$, it is a point at which, for some value of n, $f_n(x) > H$. Hence the set of points at which $f(x) > H$ is a subset of the generalised outer limiting set of the sets of points at which for fixed n, $f_n(x) > H$, as n assumes all positive integral values. *A fortiori* the set of points at which the sequence $f_1, f_2, \ldots$ diverges to $+\infty$, is a sub-set of that outer limiting set, so that the content of that outer limiting set is greater than I.

But the outer content of a generalised outer limiting set is the unique limit of the outer contents of the defining sets.† Therefore the limit of the content of the set of points at which $f_n(x) > H$ is greater than I, whence the required result at once follows.

COR.—*If $f_1, f_2, \ldots$ is a monotone ascending sequence of essentially positive summable functions, and the sequence of integrals has for limit a finite function $F(x)$, then the original sequence converges, except possibly at a set of content zero, and is integrable term-by-term.*‡

For, since the functions are never negative,

$$\int_a^b f_n(x)\,dx > HJ_n,$$

where J_n denotes the content of the set of points at which $f_n(x) > H$. Also, since the sequence is monotone increasing,

$$F(b) - F(a) \geqslant \int_a^b f_n(x)\,dx.$$

Hence J_n is less than $[F(b) - F(a)]/H$, and therefore, by the theorem, the content of the points of divergence $\leqslant [F(b) - F(a)]/H$, which, if H is sufficiently large, is as small as we please.

This proves that the f-sequence converges except at a set of content zero.

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* In the present paper a sequence of functions is to be understood to mean a succession whose upper and lower functions agree everywhere, whether the common values are all finite or not.

† Young's *Theory of Sets of Points*, p. 104.

‡ G. Vitali, "Sull' Integrazione per Serie," 1907, *Rend. di Palermo*, Vol. XXIII, p. 151.

Hence, if at every point of this exceptional set we change the value of each $f_n(x)$ to zero, the new sequence will converge everywhere, and the limiting function $f(x)$ will differ from the upper and lower functions of the original sequence only at the exceptional set of content zero.

If now we put $g_n(x) = f_n(x)$ wherever less than A, and zero elsewhere, the g-sequence will converge to the function $g(x)$ got by the same rule from $f(x)$. Also since the g-succession is bounded, we know, by the Lebesgue theory, that $g(x)$ is summable, and that its integral is the unique limit of $\int g_n(x)\,dx$, and is therefore not greater than $F(x)$. Hence, by the definition of an integral, $f(x)$ is summable, and

$$\operatorname*{Lt}_{A=\infty} \int g(x)\,dx = \int f(x)\,dx \leqslant F(x).$$

But, since the f-succession is monotone ascending,

$$\int f(x)\,dx \geqslant \int f_n(x)\,dx,$$

whence, the sign of inequality in the preceding relation is inadmissible.* Therefore

$$\int f(x)\,dx = F(x),$$

which, since $f(x)$ only differs from the upper and lower functions of the original sequence at a set of content zero, proves the second statement of the corollary.

15. The following theorem has been given in a previous paper† in a less precise form; it is fundamental in the theory of oscillating successions.

THEOREM.—*If the integrals of a succession of functions which is bounded below (above) form a succession which nowhere diverges properly to* $+\infty$ $(-\infty)$, *then* (1) *the original succession is semi-integrable below (above), and* (2) *it oscillates finitely below (above) except at a set of content zero.*

Let $l(x)$ be the lower function of the original succession $f_1, f_2, \ldots$. Denote by w_n the function whose value at any point is the lower bound of

* This case, when the f-sequence converges, was proved by Beppo Levi.

† The first result of this theorem was proved in the case of finitely oscillating successions in my paper "On Term-by-Term Integration of Oscillating Series," 1909, *Proc. London Math. Soc.*, Ser. 2, Vol. 8, p. 111. A slight correction is needed in lines 14 and 15; they should read "integrals are finite; for, by (1), w_n is not less than s_n."

the values of $f_n, f_{n+1}, \ldots$ at that point. Then

$$w_1 \leqslant w_2 \leqslant \ldots \leqslant w_n \leqslant \ldots$$

is a monotone increasing sequence, having as limit the lower function $l(x)$. Also, since

$$w_n(x) \leqslant f_n(x),$$

and the functions f_n are summable, w_n is summable, and its integral is not greater than that of f_n. Hence

$$\operatorname*{Lt}_{n=\infty} \int_a^x w_n(x)\,dx \leqslant \text{lower } \operatorname*{Lt}_{n=\infty} \int^x f_n(x)\,dx.$$

Since the lower limit on the right is by hypothesis finite, it follows from the preceding corollary that term-by-term integration of the w-sequence is allowable, so that $l(x)$ is summable. Hence $l(x)$ is finite except at a set of content zero, whence (2) follows, and

$$\operatorname*{Lt}_{n=\infty} \int_a^x w_n(x)\,dx = \int_a^x l(x)\,dx,$$

whence (1) follows, that is

$$\int_a^x l(x)\,dx \leqslant \text{lower } \operatorname*{Lt}_{n=\infty} \int_a^x f_n(x)\,dx,$$

16. THEOREM.—*If a succession of functions is semi-integrable below (above), the upper and lower functions of the succession of integrals are upper (lower) semi-integrals, being, in fact, greater (less) than the integral of the lower (upper) function of the original succession by a monotone increasing function.*

We proceed to prove the first of these two alternative statements. Since the lower limit of the difference of two functions is not greater than the difference of the upper limits, or of the lower limits, we have, denoting the upper and lower functions of the succession of integrals by $U(x)$ and $L(x)$,

$$\text{lower } \operatorname*{Lt}_{n=\infty} \int_x^{x+h} f_n(x)\,dx \leqslant U(x+h)-U(x),$$

and also

$$\leqslant L(x+h)-L(x).$$

From the former inequality, since the succession is semi-integrable below,

$$\int_x^{x+h} l(x)\,dx \leqslant U(x+h)-U(x),$$

whence

$$\int_a^{x+h} l(x)\,dx - U(x+h) \leqslant \int_a^x l(x)\,dx - U(x).$$

Similarly from the second of the above inequalities

$$\int_a^{x+h} l(x)\,dx - L(x+h) \leqslant \int_a^x l(x)\,dx - L(x),$$

which shews that the right-hand sides of both these last inequalities are monotone increasing functions, and proves the theorem in this case. Similarly the alternative theorem may be proved.

17. Theorem.—*If the points* in the neighbourhood of which $f_n(x)$, regarded as a function of the ensemble (n, x), is unbounded above (below) form at most a set of zero content, then the necessary and sufficient condition that the f-succession should be semi-integrable above (below) is that the upper (lower) function of the succession of integrals should be a lower (upper) semi-integral.*

That this condition is necessary has just been proved (§ 16), that it is sufficient follows easily. For, since the exceptional points form a closed set and it is, by hypothesis, of zero content, it can be enclosed in a finite number of intervals D, the sum of whose lengths is as small as we please.

Then, by the characteristic properties of an integral and a lower semi-integral, the integral of the upper function of the f-succession over these intervals is as small as we please numerically, and the positive increment of the upper function of the integral succession is also as small as we please. That is,

$$-e \leqslant \int_D u(x)\,dx \leqslant e,$$

$$\text{upper } \underset{n=\infty}{\text{Lt}} \int_D f_n(x)\,dx \leqslant e.$$

But in each of the complementary intervals d the f-succession is bounded and therefore semi-integrable above, that is

$$\text{upper } \underset{n=\infty}{\text{Lt}} \int_d f_n(x)\,dx \leqslant \int_d u(x)\,dx.$$

Hence, since the upper limit of a sum is $\leqslant$ the sum of the upper limits,

$$\text{upper } \underset{n=\infty}{\text{Lt}} \left\{ \int_{D+d} f_n(x)\,dx \right\} \leqslant \int_{D+d} u(x)\,dx + 2e.$$

Since e is at our disposal, this proves the theorem.

Cor.—*If the points in the neighbourhood of which $f_n(x)$, regarded as a*

* These points are the points at which the peak (chasm) function of the succession has the value $+\infty$ $(-\infty)$.

function of the ensemble (n, x) *is unbounded form a set of content zero, the necessary and sufficient condition that the succession should be semi-integrable both above and below is that the lower and upper functions of the succession of integrals should be integrals.*

This may be put another way:—we may say that, *if the points at which the peak and chasm functions are infinite form a set of zero content, the succession will be semi-integrable both above and below if, and only if, all the upper functions and all the lower functions of all possible sub-successions are integrals. Moreover every unique limiting function of a sub-sequence, if such functions exist, is accordingly an integral.*

The following example shews that the restriction imposed on the content of the points at which the succession is unbounded is essential. We have here a succession which converges everywhere to zero, while the succession of integrals converges to the unique limiting function x, which is an integral.

18. *Standard Example of a Succession which is not Semi-Integrable above.**

Divide the interval (0, 1) into 2^n equal parts, and in each of these define $h_n(x)$ by the same rule, viz., divide that segment into 2^n parts, and let $h_n(x)$ be $= 2^n$ at every point of the most left-hand part, and be zero at all the remaining points. (Fig. 1.)

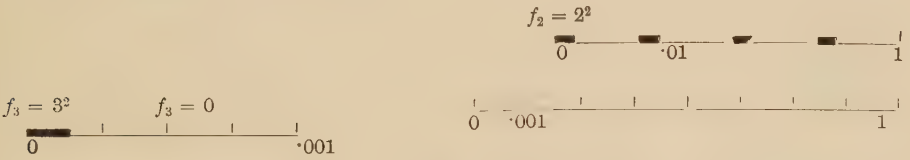

Fig. 1.

Then the sum of the lengths of the intervals in which $h_n(x) = 2^n$ is 2^{-n}; therefore the sum of the lengths of all the intervals in which any of the functions from and after $h_n(x)$ are positive is 2^{1-n}; and therefore the content of those intervals is less than 2^{1-n}, and has zero as limit, when n increases indefinitely.

Hence† the points belonging to an infinite number of these sets of

* Cp. Vitali, "Sull' Integrazione per Serie," 1906, *Rend. di Palermo*, Vol. xxiii, p. 155.

† Young's *Theory of Sets of Points*, Theorem 19, p. 96.

intervals form a set E of content zero, which includes, of course, all the binary points of division.

At the points of E the upper function of the h-succession is therefore infinite, and elsewhere it is zero. The lower function is always zero.

If now at all the points of E we put $f_n(x) = 0$, and elsewhere $f_n(x) = h_n(x)$, it is evident that *the f-succession will converge everywhere to zero.* But

$$\operatorname*{Lt}_{n=\infty} \int_0^x f_n(x)\,dx = \operatorname*{Lt}_{n=\infty} \int_0^x h_n(x)\,dx = x.$$

Thus the integral of the upper function

$$< \operatorname*{Lt}_{n=\infty} \int_0^x f_n(x)\,dx.$$

*Standard Example of an Oscillating Succession which is Semi-Integrable above without being Integrable above.**

19. Let all the functions of the g-succession be zero except in the intervals now to be specified.

$g_1 = 1$ in the interval (·1, 1), using the binary notation, that is $(\frac{1}{2}, 1)$;

$g_2 = 1$,, (0, ·1);

$g_3 = 1$,, (·11, 1);

$g_4 = 1$,, (·1, ·11);

$g_5 = 1$,, (·01, ·1);

$g_6 = 1$,, (0, ·01),

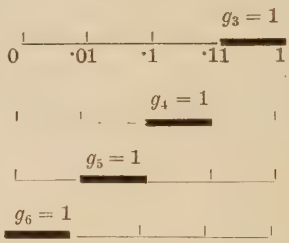

FIG. 2.

(Fig. 2), and so on; generally, if $n = 2^r - 1$, the $n+1$ functions

$$g_n,\ g_{n+1},\ \ldots,\ g_{2n},$$

are each $= 1$ in one of the $(n+1)$ equal parts into which we divide the segment (0, 1), these segments being assigned to the functions in order from right to left. In this way *the upper function of the succession is clearly* 1 *and the lower function* 0 *everywhere. The integral of the constituent function g_n, however, decreases without limit.* Thus, as of course

* That is to say, without the upper limit of the integrals being the integral of the upper function.

must be the case, since the succession is bounded below and the integrals form a sequence, the integral of the lower function

$$= \operatorname*{Lt}_{n=\infty} \int_0^x g_n(x)\,dx,$$

but the integral of the upper function

$$= x > \operatorname*{Lt}_{n=\infty} \int_0^x g_n(x)\,dx.$$

20. THEOREM.—*If the points in the neighbourhood of which $f_n(x)$, regarded as a function of the ensemble (n, x), is unbounded above (below) form at most a set of zero content, the f-succession is semi-integrable above (below), provided a g-succession which is semi-integrable above (below) can be found, such that, for all values of n, $f_n(x)$ is less (greater) than or equal to $g_n(x)$.*

For, if the g-succession is semi-integrable above, the upper limit of the integrals is, by § 16, a lower semi-integral. Also, since

$$f_n(x) \leqslant g_n(x),$$

$$\int_a^x f_n(x)\,dx = \int_a^x g_n(x)\,dx$$

—a positive monotone increasing function;

and therefore $\text{upper } \operatorname*{Lt}_{n=\infty} \int_a^x f_n(x)\,dx = \text{upper } \operatorname*{Lt}_{n=\infty} \int_a^x g_n(x)\,dx$

—a positive monotone increasing function.

Thus the upper limit of the integrals of the f-succession is also a lower semi-integral, whence, by § 17, the f-succession is semi-integrable above.

Similarly the alternative theorem may be proved.

COR.—*If the points in the neighbourhood of which $f_n(x)$, regarded as a function of the ensemble (n, x), is unbounded, form at most a set of zero content, the f-succession is semi-integrable both above and below, provided two successions can be found, whose constituents are respectively $\geqslant$ and $\leqslant$ the corresponding constituents of the f-succession, the former of these successions being semi-integrable above and the latter semi-integrable below.*

It may be remarked that the tests here given are necessary as well as sufficient, on the assumption that the exceptional points do not form a set of positive content.

If the exceptional points of the auxiliary successions have zero content, those of the f-succession have, of course, zero content.

21. It will be seen that *the restriction imposed, by which the points at which the succession is unbounded above (below) form a set of content zero, is essential.* In fact, if we take the standard example, given in § 18, of an f-succession which is not semi-integrable above, and add to each constituent function $f_n(x)$ the corresponding constituent $g_n(x)$ of the standard g-succession semi-integrable above given in § 19, the succession f_1+g_1, f_2+g_2, ... is semi-integrable above. For, since the upper function of the f-succession is identically zero, the upper function of the $(f+g)$-succession is the same as that of the g-succession, so that its integral from 0 to x has the value x. Since the integral of $f_n(x)$ has the limit x, and the value of the integral of $g_n(x)$ decreases with zero as limit as n increases, the limit of the integral of $[f_n(x)+g_n(x)]$ is x, and is equal to the integral of the upper function of the $(f+g)$-succession, so that this latter succession is, as was stated, semi-integrable above. Moreover, since $g_n(x)$ is positive, the constituents of the $(f+g)$-succession are greater than or equal to the corresponding constituents of the f-succession.

This proves that, if we omit the restriction as to the content of the exceptional points being zero, the preceding theorem is longer true.

It may be remarked that the $(f+g)$-succession, though semi-integrable both above and below, is not integrable;* in fact, as is proved below (§ 26), if the $(f+g)$-succession were integrable, the f-succession would necessarily be semi-integrable above, whether or not the exceptional points were of zero content.

22. The property of a succession of being semi-integrable above (below) relates to the integrals taken over any the same segment; if we integrate over any set of points, and the characteristic inequality is still true, irrespective of the particular set chosen, the succession is said to be *absolutely semi-integrable above (below)*. This distinction is only of importance when the succession is unbounded, as the following theorem shews:—

THEOREM.—*If the f-succession is bounded above (below), it is absolutely semi-integrable above (below).*

For, let S be any set of points, and g_n the function equal to f_n at every

* See § 13.

point of S and zero elsewhere, while v is the function got by the same rule from the upper function $u(x)$. Then the upper function of the g-succession is $v(x)$, and the g-succession is bounded above, if the f-succession is so.

Hence, by the theory of oscillating successions (§ 15),

$$\operatorname*{Llt}_{n=\infty} \int_a^b g_n(x)\,dx \leqslant \int_a^b v(x)\,dx,$$

that is,

$$\operatorname*{Llt}_{n=\infty} \int_S f_n(x)\,dx \leqslant \int_S u(x)\,dx.$$

Since this is true for every set S, this proves the first of the alternative theorems. Similarly the second may be proved.

23. THEOREM.—*If $f_1(x), f_2(x), \ldots$ is a succession of summable functions,* having a summable upper (lower) function, then the succession is absolutely semi-integrable above (below) provided*

$$\int_E f_n(x)\,dx,$$

regarded as a function of the ensemble (n, E), has no positive (negative) double limit, when n increases indefinitely and E approaches zero.

For, if G_k denote the set of points at which at least one of the functions $f_n(x)$ is greater than 2^k, the set G of all the points common to all the sets G_k (which, as is evident, are such that each contains the next set after it), consists of the points at which at least one of the functions f_n is $+\infty$, and of the points at which the upper function $u(x)$ is $+\infty$.

Since the functions f_n and u are summable, each of these sub-sets is of content zero, and therefore G is itself of content zero. Hence the content of G_k has, as k increases indefinitely, zero as limit. Thus we may denote the content of G_k by E, and we then have, by hypothesis,

$$\operatorname*{Llt}_{k=\infty,\, n=\infty} \int_E f_n(x)\,dx \leqslant 0. \tag{1}$$

Let g_n denote the function which $= f_n$ except at the points of G_k, where it is zero, and let v be the function got by the same rule from $u(x)$. Then v is evidently the upper function of the succession of functions $g_n(x)$, which is bounded above, so that, by the preceding theorem, it is absolutely semi-integrable above.

* These functions may be infinite, but of course only at a set of content zero, and the succession need not oscillate finitely.

Now, by the definition of g_n, the integral of g_n over any set S is the integral of f_n over the part of S not belonging to G_k. Hence, denoting the set of the common points of S and G_k by (S, E), we have

$$\int_S f_n(x)\,dx = \int_S g_n(x)\,dx + \int_{(S,\,E)} f_n(x)\,dx,$$

and

$$\int_S u(x)\,dx = \int_S v(x)\,dx + \int_{(S,\,E)} u(x)\,dx.$$

Hence, since the g-succession is absolutely semi-integrable above,

$$\text{upper } \operatorname*{Lt}_{n=\infty} \left[\int_S f_n(x)\,dx - \int_{(S,\,E)} f_n(x)\,dx\right] \leqslant \int_S u(x)\,dx - \int_{(S,\,E)} u(x)\,dx.$$

By the fundamental property of an integral, the second integral on the right has zero as unique limit as k increases indefinitely, and in consequence, E and (S, E) diminish indefinitely. Also, by (1), all the double limits of the second integral on the left are negative or zero, so that the same is true of the repeated limits; hence the second integral on the left may be omitted without affecting the inequality which we then obtain, when we let k increase indefinitely,

$$\text{upper } \operatorname*{Lt}_{n=\infty} \int_S f_n(x)\,dx \leqslant \int_S u(x)\,dx.$$

which shews that the f-succession is absolutely semi-integrable above. This proves the first of the alternative theorems; similarly the other may be proved.

COR.—*If both the upper and the lower function are summable, and* $\int_E f_n(x)\,dx$ *has the unique double limit zero, the f-succession is absolutely semi-integrable both above and below.*

If, in addition, the upper and lower functions agree except at a set of content zero the f-succession may be integrated term-by-term over every set of points, that is,*

$$\operatorname*{Lt}_{n=\infty} \int_S f_n(x)\,dx \leqslant \int_S u(x)\,dx = \int_S l(x)\,dx.$$

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* This would at the first glance seem to be in contradiction to a theorem of Vitali's, *loc. cit.*, p. 139, but this is only due to the fact that in the statement, not in the proof, of his theorem, Vitali has included the convergence of the series among his necessary and sufficient conditions for term-by-term integration.

The only case considered by Vitali is that of a sequence of functions, and he accordingly uses terms depending on this restriction. Thus the term "integrabile completamente per serie," used by him, implies that, *whatever set we integrate over, the succession of integrals*

24. THEOREM.—*If the upper (lower) function of a succession of summable functions is itself a summable function, and the succession is absolutely semi-integrable above (below), then*

$$\int_E f_n(x)\,dx$$

has, when n increases indefinitely and E decreases with zero as limit, no positive (negative) double limit.

In the first place it is obvious that, if the succession is bounded above, none of the double limits can be positive; for, if for all values of n, $f_n < 2^k$, the integral whose limits are considered is less than $E2^k$, which has zero as limit when E has zero as limit, however n may vary.

If, then, the f_n's be unbounded above, and g_n be $= f_n$, except at the points of the set G_k where it is zero, $\int_E g_n(x)\,dx$ will have no positive double limit. Here, as in a previous proof, G_k denotes the set of all the points at which at least one of the functions f_n is greater than 2^k, and, as before, the content of G_k has zero as limit, when k is indefinitely increased. Hence, by the fundamental property of an integral, so often used in the present paper, we can determine k so that the integral of $u(x)$

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converges, and has as limit the integral of the unique limiting function of the sequence. This is a special case of the circumstances contemplated in the above corollary.

Vitali's condition is formally different, but actually the same, as that given above. He demands that the integrals of the sequence should be what he calls "assolutamente continui," that is to say, *pre-assigning any positive quantity e, he requires that it should be possible to determine a positive quantity E_0, such that, provided only the content of the set E, over which we propose to integrate, is less than E_0, $\int_E f_n(x)\,dx$ is less than e.*

Whatever succession of ensembles (E, n), we choose, n increasing indefinitely and E approaching zero as limit, there will then be a stage from and after which all the sets considered have content less than E_0, so that all the double limits of $\int_E f_n(x)\,dx$ are numerically less than e. Since this is true for all choices of e, the integral considered has zero as unique double limit. Thus *Vitali's condition being satisfied, my condition is so also.*

On the other hand, if my condition is satisfied, so that $\int_E f_n(x)\,dx$ has zero as unique double limit, we can assign e and then determine E_0' and m, so that, for all integers n greater than m, and all sets E whose content is less than E_0' the integral considered is less than e. But, by the fundamental property of an integral so often used in the present paper, we can find E_i so that, provided the content of the set E is less than E_i, $\int_E f_i(x)\,dx$ is less than e. Thus, if we choose E_0 to be the least of the quantities E_0', E_1, E_2, ..., E_m, $\int_E f_n(x)\,dx$ is, for all values of n, less than e. Thus *my condition being satisfied, Vitali's is so also.*

over the set G_k, *or any of its sub-sets*, is less than e, where e is any pre-assigned positive quantity. Thus

$$\int_{G_k} u(x)\,dx < e\,;$$

and therefore, since the succession is absolutely semi-integrable above,

$$\text{upper } \underset{n=\infty}{\text{Lt}} \int_{G_k} f_n(x)\,dx < e.$$

We can, in consequence, determine m, so that for all values of n greater than m,

$$\int_{G_k} f_n(x)\,dx < e.$$

Now, if E_k denote the common part of G_k and the set E over which we propose to integrate, $f_n(x)$ is certainly positive at every point of G_k, and therefore

$$\int_{E_k} f_n(x)\,dx \leqslant \int_{G_k} f_n(x)\,dx \leqslant e,$$

whence $$\int_E f_n(x)\,dx = \int_E g_n(x)\,dx + \int_{E_k} f_n(x)\,dx < \int_E g_n(x)\,dx + e.$$

Since this inequality holds for all values of n greater than m, and for the fixed value of k chosen and E has been chosen arbitrarily, we have, remembering that all the double limits of the integral on the right are $\leqslant 0$,

$$\underset{n=\infty,\,E=0}{\text{Llt}} \int_E f_n(x)\,dx < e.$$

Since this is true for all positive values of e, it shews that the integral on the left has no positive double limit, when n increases indefinitely, and E has zero as limit.

25. From the preceding theorem we have immediately the following :—

THEOREM.—*If a succession of functions is absolutely semi-integrable above* (*below*), *the succession of integrals oscillates uniformly and homogeneously above* (*below*).*

For $$\int_a^{x+h} f_n(x)\,dx = \int_a^x f_n(x)\,dx + \int_x^{x+h} f_n(x)\,dx,$$

and, by the preceding theorem, the latter integral on the right has no

* In this case all the upper functions and all the lower functions of the succession of integrals or any of its sub-successions are upper (lower) semi-continuous functions. Cp. W. H. Young, "On Homogeneous Uniform Oscillation," 1910, *Proc. London Math. Soc.*, Vol. 8, pp. 353–364.

positive double limit, when n increases indefinitely and h has zero as limit. But the upper double limit of the integral on the left is the peak function $\Pi(x)$ of the succession of integrals, and the upper limit of the first integral on the right is the upper function $U(x)$ of its succession of integrals; thus

$$\Pi(x) \leqslant U(x).$$

But the peak function is never less than the upper function, therefore

$$\Pi(x) = U(x),$$

that is to say, the succession of integrals oscillates uniformly above. Moreover, the necessary and sufficient condition in order that a succession may be absolutely integrable above is a homogeneous condition, that is to say, it applies equally to the whole succession and to any of its sub-successions. Hence the above applies equally to the succession and to any of its sub-successions, so that any sub-succession of the succession of integrals oscillates uniformly above, that is, the succession of integrals oscillates uniformly and homogeneously above.

26. The necessary and sufficient condition that a succession of functions should be absolutely semi-integrable above (below), proved in the two Theorems of §§ 23, 24, viz., that $\int_E f_n(x)\,dx$ should have no positive (negative) double limit, enables us to see that *a succession of positive functions which is semi-integrable above is not necessarily absolutely semi-integrable above.*

For, if the condition above referred to holds when f_n is positive, it obviously holds for another succession of positive functions, $g_1, g_2, \ldots$, where $g_n \leqslant f_n$. Hence the g-succession would be also semi-integrable above absolutely, and therefore certainly semi-integrable above ordinarily. We have, however, given an example which contradicts this statement, namely, that in § 21; the two successions in question are there called the $(f+g)$-succession, and the f-succession. Of these the former is a succession of positive functions which is semi-integrable above, and the latter a smaller succession of positive functions which is not semi-integrable above.

Hence, also, *a succession of functions which is bounded below (above) and semi-integrable above (below), is not necessarily absolutely semi-integrable above (below).*

We have, however, the following theorem, already referred to in § 21.

THEOREM.—*If a succession of functions is bounded below (above) and*

integrable term-by-term, the succession is necessarily absolutely integrable, that is to say, the integrals of the upper and lower functions over any set of points are equal to one another and to the unique limit of the integrals of the constituent functions of the succession over the set of points considered.*

For a succession which is bounded below is, as we saw (§ 22), absolutely semi-integrable below. Hence, denoting by S any set of points, and by T the complementary set, we have, in this case,

$$\operatorname*{Lt}_{n=\infty} \int_S f_n(x)\,dx \geqslant \int_S l(x)\,dx,$$

$$\operatorname*{Lt}_{n=\infty} \int_T f_n(x)\,dx \geqslant \int_T l(x)\,dx.$$

But, by hypothesis, $$\operatorname*{Lt}_{n=\infty} \int_a^b f_n(x)\,dx = \int_a^b l(x)\,dx,$$

which necessitates each of the two above inequalities being an equality, and so proves the theorem.

27. THEOREM.—*If a summable function $g(x)$ can be found, such that, for all values of n and x,*

$$f_n(x) \leqslant g(x) \text{ [or, in the second case, } g(x) \geqslant f_n(x)],$$

the f-succession is semi-integrable above (below).

For, let $w_n(x)$ denote the function whose value at each point x is the upper bound of the values of $f_n, f_{n+1}, f_{n+2}, \ldots$ there, then, it follows from the hypothesis that, for all values of n,

$$f_n(x) \leqslant w_n(x) \leqslant g(x); \tag{1}$$

also the functions w_n form a monotone descending sequence, so that we may integrate term-by-term, and, since their limit is the upper function $u(x)$ of the f-succession,

$$\operatorname*{Lt}_{n=\infty} \int w_n(x) = \int u(x)\,dx.$$

But, by (1), $$\int f_n(x) \leqslant \int w_n(x);$$

therefore $$\operatorname*{Llt}_{n=\infty} \int f_n(x)\,dx \leqslant \int u(x)\,dx,$$

which proves the theorem.

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* See § 13.

COR.—*Under these circumstances the f-succession is also absolutely semi-integrable above (below).*

In fact, if we replace the values of all the functions concerned at the complementary set of any chosen set S by zero, the above reasoning holds good of the new functions, and the result may be written

$$\operatorname*{Llt}_{n=\infty} \int_S f_n(x)\,dx \leqslant \int_S u(x)\,dx.$$

Alternative proof.—The above proof is independent of the considerations we have been adducing. It is evident, however, that the theorem is an immediate consequence of the fact that a succession of functions is certainly integrable above if its integrals are less by a positive monotone increasing function than a function which is an integral; for the necessary and sufficient condition for semi-integrability above is satisfied. The theorem is, in fact, a particular case of the following theorem.

28. THEOREM.—*A succession of functions is absolutely semi-integrable above (below) if another succession which is absolutely semi-integrable above (below) can be found, whose constituent function is $\geqslant (\leqslant)$ the corresponding constituent function of the given succession.*

Here again the condition for absolute semi-integrability above (below) given in § 23 is obviously satisfied.

As in practice there may be a difficulty in determining whether a succession which is semi-integrable above is so absolutely, the following special case of this theorem rises in importance:—

COR.—*A succession of functions $f_1, f_2, \ldots$ is absolutely semi-integrable above (below) if an integrable succession $g_1, g_2, \ldots$ can be found, which is bounded below (above), such that $g_n(x)$ is greater (less) than or equal to $f_n(x)$, for all values of n and x.*

It will be remarked that the condition that the g-succession should be absolutely semi-integrable above (below) is essential in this reasoning. We cannot, therefore, omit the boundedness below of the g-succession in the corollary. The reasoning, moreover, by which we proved the theorem of § 27 does not apply here, since it does not follow because the g-succession is absolutely semi-integrable above (below) that there exists a summable function which is $\geqslant (\leqslant)\ g_n(x)$ for all values of n.

The limitation that the g-succession must be bounded below is not one which in practice is likely to cause much inconvenience. If, however, the

succession whose semi-integrability is under discussion is not absolutely semi-integrable, the test is necessarily inapplicable; as it, however, constitutes a necessary and sufficient condition for the absolute semi-integrability above of the f-succession, it must succeed, except in the rare case in question. It is, in fact, obvious, that a succession which is absolutely semi-integrable above, remains so when we replace all the values less than $-K$ by the value $-K$.

29. It remains to give a test applicable in the exceptional case. This we are only able to do when it is either known *a priori*, or ascertainable, that the points in the neighbourhood of which the succession is unbounded above (below) form a set of zero content. We then have the theorem given in § 17.

If it be further known that the set of points in question is not only of zero content, but countable, we have the theorem given in my paper on "Term-by-Term Integration of Oscillating Series." *It is sufficient to shew that the upper function of the succession of integrals is lower semi-continuous on the left and upper semi-continuous on the right to be sure that semi-integration above is allowable.* This is a test which is very convenient to apply in practice; it is even more convenient in many cases than the test by comparison, and it is applicable whether the given succession is as a matter of fact absolutely semi-integrable or not; it constitutes, moreover, a necessary as well as sufficient test.

30. The tests we have given have all been of a type which permitted of the separate consideration of semi-integration above and below, and the wide character of their applicability as well as the conciseness of the proofs is largely due to this characteristic. A test of a different character which deals simultaneously with semi-integrability above and below is the following, and is in certain cases very useful. It has been stated and proved by Lebesgue* for sequences in a slightly different form.

THEOREM.—*The succession*

$$f_1(x)\,g(x),\ f_2(x)\,g(x),\ \ldots,$$

is absolutely semi-integrable both above and below, if the upper and lower functions of the succession are summable, if g^2 is summable, and if further

* H. Lebesgue, "Sur les intégrales singulières," p. 50.

$\int f_n^2\,dx$ is, for all values of n and all sets over which we integrate, less than a finite constant, say B.

For, by Schwarz's lemma,*

$$\left(\int_E f_n g\,dx\right)^2 \leqslant \int_E f_n^2\,dx \int_E g^2\,dx \leqslant B\int_E g^2\,dx.$$

This shews that the left-hand side has the unique double limit zero when n increases without limit, and E decreases to zero, which, by § 23, proves the theorem.

31. We have seen that the theory of semi-integrability above (below) is less simple than that of absolute semi-integrability above (below). In the only theorems that we have been able to give we have had to predicate that a certain set of points should in one form of test be of zero content, and, in another form, narrower but more easily applied, that this set should be countable. It is very remarkable, however, that, when we come to the theory of integrable sequences, the theory simplifies itself considerably. We have, in fact, the following necessary and sufficient test for the integrability of a sequence.

THEOREM.†—*The necessary and sufficient condition that a sequence*

$$f_1,\ f_2,\ \ldots,\ f_n,\ \ldots$$

should be integrable is that it should be possible to find two other sequences, each of which is integrable, and such that

$$h_n \leqslant f_n \leqslant g_n.$$

For the functions $g_n - h_n$ form a sequence except at most at a set of content zero, and each constituent function of this sequence, if we make a suitable convention as to its value at the points where g_n and h_n are infinite, is a positive function. Hence this sequence, since it is certainly integrable, is absolutely integrable. But

$$0 \leqslant f_n - h_n \leqslant g_n - h_n.$$

Hence the sequence of functions $f_n - h_n$ is absolutely integrable. But the h-sequence is, by hypothesis, integrable, therefore the sequence of func-

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* Id., p. 37; see also W. H. Young, "On a New Method in the Theory of Integration."

† See elow, § 34.

tions $h_n+(f_n-h_n)$ is integrable, which proves that the condition is sufficient. That it is necessary is obvious, for we only have to put

$$g_n = \quad x+f_n,$$

$$h_n = -x+f_n,$$

and the g-sequence and h-sequence will satisfy the requisites.

COR.—*The above theorem is equally true if for sequences we substitute successions which converge except at a set of content zero,* as is clear from § 26.

32. For the convenience of readers who may be more especially interested in tests for the integration term-by-term of converging series or sequences, we collect here the more striking of such tests.

Test 1. (Osgood-Lebesgue.)—*If the sequence of functions is in its entirety bounded both above and below, term-by-term integration is always allowable.*

Test 2.—*A monotone sequence of functions, each of which is summable, may always be integrated term-by-term.*

Test 3.—*If a sequence of functions is bounded below, then, provided another sequence can be found which is integrable term-by-term and is such that its constituent function is greater than or equal to the corresponding constituent of the given sequence, term-by-term integration of the original sequence is allowable.*

A similar test may be used interchanging "below" with "above," and "greater" with "less."

Test 4. (This is a special case of Test 3.)—*If a sequence of functions is bounded below (above), then, provided a summable function can be found which is greater (less) than or equal to every function of the sequence, term-by-term integration is allowable.*

Test 5.—*If two summable functions $g(x)$ and $h(x)$ can be found, such that the constituent function of the sequence lies at every point between $g(x)$ and $h(x)$, then term-by-term integration is allowable.*

Test 6.—*If two integrable sequences of functions $g_n(x)$ and $h_n(x)$ can*

be found, such that

$$g_n(x) \leqslant f_n(x) \leqslant h_n(x),$$

for all values of n and x, term-by-term integration of the f-sequence is allowable.

Test 7. (Osgood.)—*If the points in the neighbourhood of which the sequence of functions is unbounded are at most countable, then the sequence is integrable term-by-term provided only the integrals of the sequence converge to a continuous limiting function.*

It should be noticed that Tests 3, 4, 7 are, *under the circumstances stated*, necessary as well as sufficient. Test 6 is always necessary and sufficient. It should also be noticed that one test may be utilised in the application of another; thus Test 2 can be utilised in giving a special form to other tests, *e.g.*, we have the following:—

Test 8.—*If the constituent function of a sequence lies between the corresponding constituents of two monotone sequences of summable functions, whose limiting functions are summable, term-by-term integration is always allowable.*

We note that the integrability of a monotone sequence is always absolute, and hence, also, that it follows that the given sequence is not merely integrable, but absolutely integrable.

In fact we have the following test for absolute integrability:—

Test 9.—*If a g-sequence and an h-sequence can be found, each of which is absolutely integrable, and for all values of n and x,*

$$g_n(x) \leqslant f_n(x) \leqslant h_n(x),$$

then absolute term-by-term integration of the f-sequence is allowable.

The following test, though of a very special character, may in certain cases be applied with great ease.

Test 10. (Lebesgue.)—*The sequence* $f_1g, f_2g, \ldots, f_ng, \ldots$ *is integrable term-by-term if both its limiting function and* g^2 *are summable functions, and if* $\int f_n^2\,dx$ *over the interval considered is for all values of n less than a finite constant.*

Another form of the same test is the following:—

The sequence in question is integrable term-by-term if the limiting function and g^2 *are both summable, and* $\int (f_n - f)^2\,dx$ *over the interval considered is for all values of n less than a finite constant.*

The following theorems which can be utilised as tests are worth noting :—

Theorem.—*If two sequences of positive functions* $f_1, f_2, \ldots$ *and* $g_1, g_2, \ldots$ *are integrable term-by-term, the sequence whose constituent function is* $f_n g_n$ *is integrable term-by-term.*

Theorem.—*Under the same circumstances the sequence whose constituent function is* h_n, *where* h_n *is equal at each point to the greater of* f_n *and* g_n, *is also integrable term-by-term.*

The first of these theorems follows from the second by means of Test 3. It suffices therefore to prove the latter theorem.

We have merely to remark that the succession whose constituent is (f_n+g_n) is certainly integrable ; the required result follows then by Test 3.

Theorem.—*If the sequence whose constitutent is* f_n^2 *is integrable term-by-term, so is the f-sequence.*

Theorem.—*If the sequence whose constituent is* $|f_n|$ *is integrable term-by-term, so is the f-sequence.*

Both these theorems follow from Test 6, in its two portions. To prove the second, put

$$g_n(x) = -|f_n(x)| \quad \text{and} \quad h_n(x) = +|f_n(x)|.$$

To prove the first, take $h_n(x) = f_n^2$, wherever f_n is greater than 1, and $= f_n$ at the remaining points, and $g^n = -f_n^2$, wherever f_n is less than -1, and $= f_n$ at the remaining points.

33. So far we have tacitly assumed that the limits of integration are finite. It is plain that, when this is not the case, the reasoning of the paper will not always hold as it stands, and many of the results will require modification. We content ourselves here with shewing that the very important test, obtained in § 31, still holds, when the interval over which we integrate is an infinite interval. For this purpose it will be convenient first to prove the following inequality between the repeated limits of a function of two variables, which is monotone with respect to one of them :—

Lemma.—*If* $F(X, n)$ *is a monotone increasing function of* X,

$$\operatorname*{Lt}_{X=\infty} [\text{lower } \operatorname*{Lt}_{n=\infty} F(X, n)] \leqslant \text{lower } \operatorname*{Lt}_{n=\infty} [\operatorname*{Lt}_{X=\infty} F(X, n)].$$

For, since $F(X, n)$ increases with X,

$$F(X, n) \leqslant \underset{X=\infty}{\text{Lt}}\, F(X, n);$$

therefore $$\text{lower } \underset{n=\infty}{\text{Lt}}\, F(X, n) \leqslant \text{lower } \underset{n=\infty}{\text{Lt}} \left[\underset{X=\infty}{\text{Lt}}\, F(X, n)\right],$$

whence the required result at once follows.*

THEOREM.—*If $f_1, f_2, \ldots$ is a succession of positive functions, each of which is summable throughout the whole infinite interval $(0, \infty)$, then the succession is semi-integrable below throughout that interval, whether, or not, the succession is bounded above.*

Denoting by $l(x)$ the lower function of the f-succession, we prove, as before, that

$$\int_0^X l(x)\,dx \leqslant \text{lower } \underset{n=\infty}{\text{Lt}} \int_0^X f_n(x)\,dx.$$

Now, since $f_n(x)$ is positive, the integral on the right is a monotone function of its upper limit; hence, by the lemma,

$$\underset{X=\infty}{\text{Lt}} \left[\text{lower } \underset{n=\infty}{\text{Lt}} \int_0^X f_n(x)\,dx\right] \leqslant \text{lower } \underset{n=\infty}{\text{Lt}} \left[\underset{X=\infty}{\text{Lt}} \int_0^X f_n(x)\,dx\right].$$

Therefore $$\int_0^\infty l(x)\,dx \leqslant \text{lower } \underset{n=\infty}{\text{Lt}} \left[\int_0^\infty f_n(x)\,dx\right].$$

34. From the above theorem we at once get the following:—

THEOREM.—*If $f_1, f_2, \ldots$ be a sequence of positive functions, which is integrable throughout the whole infinite interval $(0, \infty)$, and $g_1, g_2, \ldots$ is any sequence of positive functions, such that $g_n \leqslant f_n$, the g-sequence is also integrable throughout the whole infinite interval.*

In fact, denoting the excess of f_n over g_n by h_n, the h's will, like the g's, form a sequence, to which the preceding theorem applies, for plainly the g's and the h's will all be summable.

Adding then the two inequalities for the g's and the h's respectively,

* Similarly, of course, the same inequality holds if on both sides we change "lower" into "upper."

we must get an equality, which proves that both the inequalities are themselves equalities.

The reasoning of § 31 applies therefore as it stands to this case, for it depends precisely on the theorem for a finite interval which we have just shewn to hold good for an infinite interval. Hence the theorem stated in that article remains true when the interval under consideration is infinite.

It is also of interest to remark that the reasoning by which this result has been proved, like that of § 33, is unaffected if the integrals considered are non-absolutely convergent integrals. This follows from the fact that $h_n - g_n$ and $f_n - g_n$ are positive functions for all values of n.

35. The theorems we have obtained in the theory of oscillating series, may obviously be applied to obtain corresponding theorems in the theory of integrals. Thus we have the following important theorem in parametric integration :—

Theorem.—*If $f(x, y)$ is for each fixed value of y continuous with respect to y, except for a set of values of x of zero content, and if $g(x, y)$, $h(x, y)$ be two other functions possessing the same property, then, without any hypothesis as to the boundedness of these functions, we can assert that $\int f(x, y)\,dx$ is a continuous function of y, if $\int g(x, y)\,dx$ and $\int h(x, y)\,dx$ are so, provided only*

$$g(x, y) \leqslant f(x, y) \leqslant h(x, y).$$

This theorem has a number of important consequences, analogous to the corresponding ones for series. In particular we have the following theorem, which I had already proved elsewhere* by another method :—

Theorem.—*Under the same circumstances as to the continuity with respect to y of the function $f(x, y)$, $\int f(x, y)\,dx$ is a continuous function of y, provided $\int |f(x, y)|\,dx$ is so.*

That these theorems follow from the corresponding theorems for successions is obvious, if we reflect that, under the circumstances hypothecated,

$$\int \text{upper } \operatorname*{Lt}_{y=y_0} f(x, y)\,dx = \int \text{lower } \operatorname*{Lt}_{y=y_0} f(x, y)\,dx = \int f(x, y_0)\,dx.$$

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* W. H. Young, "On Parametric Integration," *Monatshefte für Math. und Physik*, January, 1910, pp. 1–24.

36. Again we have the following theorem in the theory of differentiation under the sign of integration :—

THEOREM.—*Differentiation under the sign of integration is allowable for* $\int f(x, y)dx$, *provided two functions* $g(x, y)$ *and* $h(x, y)$ *can be found, for which differentiation under the sign of integration is allowable, one of these functions being* $\leqslant f$, *and the other* $\geqslant f$, *and the difference in each case being a monotone increasing function of* y.

For, since $f(x, y)-g(x, y)$ and $h(x, y)-f(x, y)$ are monotone increasing functions of y,

$$g(x, y+k)-g(x, y) \leqslant f(x, y+k)-f(x, y) \leqslant h(x, y+k)-h(x, y).$$

The same is therefore true if we divide each member of this inequality by k. Hence, by the concluding test of § 33, the sequence whose general term is

$$\frac{f(x, y+k)-f(x, y)}{k},$$

if k is positive, and df/dy when k is zero, is integrable term-by-term, provided the same is true when we change f into g or into h. Now, by hypothesis,

$$\frac{d}{dy}\int g(x, y)\,dx = \int \frac{dg}{dy}\,dx,$$

that is, $$\operatorname*{Lt}_{k=0}\int \frac{g(x, y+k)-g(x, y)}{k}\,dx = \int \operatorname*{Lt}_{k=0}\frac{g(x, y+k)-g(x, y)}{k}\,dx,$$

so that the change of f into g renders the sequence integrable term-by-term. Similarly the change of f into h renders the sequence integrable. Thus

$$\operatorname*{Lt}_{k=0}\int \frac{f(x, y+k)-f(x, y)}{k}\,dx = \int \operatorname*{Lt}_{k=0}\frac{f(x, y+k)-f(x, y)}{k}\,dx,$$

that is, $$\frac{d}{dy}\int f(x, y)dx = \int \frac{df}{dy}\,dx,$$

which proves the theorem.

37. We have also the following theorem in change of order of integration in an infinite integral :—

THEOREM.—*If two functions* $g(x, y)$ *and* $h(x, y)$ *can be found, such that*

$$g(x, y) \leqslant f(x, y) \leqslant h(x, y),$$

then change of order of integration is allowable in the integrals

$$\int_0^a dx \int_0^\infty f(x, y)\, dy = \int_0^\infty dy \int_0^a f(x, y)\, dx,$$

provided it is so when the integrand is g and also when it is h, always supposing that change of order of integration is possible when the limits of integration are finite.

For, by hypothesis,

$$\int_0^a dx \int_0^\infty g(x, y)\, dx = \int_0^a dx \operatorname*{Lt}_{b=\infty} \int_0^b g(x, y)\, dy = \int_0^\infty dy \int_0^a g(x, y)\, dx$$

$$= \operatorname*{Lt}_{b=\infty} \int_0^b dy \int_0^a g(x, y)\, dx = \operatorname*{Lt}_{b=\infty} \int_0^a dx \int_0^b g(x, y)\, dy.$$

That is, denoting $\int_0^b g(x, y)\, dy$ by $G(x, b)$,

$$\int_0^a \operatorname*{Lt}_{b=\infty} G(x, b)\, dx = \operatorname*{Lt}_{b=\infty} \int_0^a G(x, b)\, dx.$$

In like manner, and with a like meaning to $H(x, b)$,

$$\int_0^a \operatorname*{Lt}_{b=\infty} H(x, b)\, dx = \operatorname*{Lt}_{b=\infty} \int_0^a H(x, b)\, dx.$$

Thus the G-sequence and the H-sequence are integrable term-by-term. Also, from the fact that $f-g$ is positive and less than $h-g$ which possesses repeated integrals, we have

$$\int_0^b dy \int_0^a f(x, y)\, dx = \int_0^a dx \int_0^b f(x, y)\, dy = \int_0^a F(x, b)\, dx,$$

where, since f lies between g and h, F lies between G and H. Therefore, since by the concluding test of § 32, the F-sequence is integrable term-by-term, we have

$$\int_0^\infty dy \int_0^a f(x, y)\, dx = \operatorname*{Lt}_{b=\infty} \int_0^a F(x, b)\, dx = \int_0^a \operatorname*{Lt}_{b=\infty} F(x, b)\, dx$$

$$= \int_0^a dx \int_0^\infty f(x, y)\, dy,$$

which proves the theorem.

COR.—*If, in addition,*

$$\int_0^\infty dx \int_0^b g(x, y)\, dy = \int_0^b dy \int_0^\infty g(x, y)\, dx,$$

for b finite and also for b infinite, and the same is true when we change g into h, then

$$\int_0^\infty dx \int_0^\infty f(x, y)\, dy = \int_0^\infty dy \int_0^\infty f(x, y)\, dx.$$

For in this case, we are given that

$$\int_0^\infty dx \operatorname*{Lt}_{b=\infty} \int_0^b g(x, y)\, dy = \operatorname*{Lt}_{b=\infty} \int_0^b dy \int_0^\infty g(x, y)\, dx$$

$$= \operatorname*{Lt}_{b=\infty} \int_0^\infty dx \int_0^b g(x, y)\, dy,$$

that is,

$$\int_0^\infty dx \operatorname*{Lt}_{b=\infty} G(x, b) = \operatorname*{Lt}_{b=\infty} \int_0^\infty G(x, b)\, dx.$$

Hence, since in this equation we may change G into H, it follows by the test that we may change G into F, which gives

$$\int_0^\infty dx \int_0^\infty f(x, y)\, dy = \operatorname*{Lt}_{b=\infty} \int_0^\infty dx \int_0^b f(x, y)\, dy,$$

which, by the theorem,

$$= \operatorname*{Lt}_{b=\infty} \int_0^b dy \int_0^\infty f(x, y)\, dx = \int_0^\infty dy \int_0^\infty f(x, y)\, dx.$$

This proves the corollary.

38. We also have the following theorem in parametric integration as an immediate corollary from the concluding theorem of § 32.

THEOREM.—*If $f(x,y)$ is a continuous function of y, and two other functions $g(x, y)$ and $h(x, y)$ can be found respectively $\geqslant$ and $\leqslant f(x, y)$ and also continuous with respect to y, or more generally if the continuity of the three functions with respect to y holds except at a set of values of x of content zero, then the continuity of $\int_0^\infty g(x, y)\, dx$ and $\int_0^\infty h(x, y)\, dx$ regarded as functions of y, carry with them the continuity of $\int_0^\infty f(x, y)\, dx$.*

Ex.—$\int_0^\infty \frac{\cos x}{a^2+x^2}\, dx$ is a continuous function of a, except when $a = 0$.

Here the larger of the two auxiliary functions g and h is $1/(a^2+x^2)$ and the smaller is $-1/(a^2+x^2)$.

We terminate this brief account of applications of the general test at the end of § 32 with the following particular test for the integration over an infinite interval which follows immediately from the general test :—

The series $u_1+u_2+\ldots$ *is integrable term-by-term from* 0 *to* ∞, *if a function* $v(x)$ *can be found, such that* $|u_n/v| < a_n$, *for all values of* n, a_n *being the general term of a convergent series of constants, and* $v(x)$ *being a function which is summable in the infinite interval in question.*

This is given here as being a generalisation of a particular test of Osgood's, in which $v(x)$ is x^{-k} and the u's are continuous functions. It is, as will be at once perceived, a very special case of our theorem.

ON THE EXISTENCE OF A DIFFERENTIAL COEFFICIENT

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1. The detailed account of the properties of derivates in Dini's *Fondamenti per la teorica delle funzioni di variabili reali* remained for some time the last word on the subject. The results there given that the upper and lower bounds of the derivates are the same in every interval, and that, in every interval, the derivates approach one another as near as we please,* might have seemed, in view of Weierstrass's † construction of a continuous function which possesses no differential coefficient anywhere of a final character. So the matter remained until Lebesgue proved that every function of bounded variation, and in particular every function with bounded derivates, possesses a finite differential coefficient everywhere, except at a set of content zero. The line of argument he adopted was in every respect a novel one. Besides the concept of the content of a set of points itself, which, though well known in the case of a closed set, required delicate constructive treatment in the case of an open set, Lebesgue made use of the still more abstract concepts of generalised integration and of Cantor induction. Neither of these latter concepts is used in the statement of the theorem, and it is not too much to say that, even at the present moment, few analysts, and still fewer mathematicians whose special interests are not analytical, are really at home with them. Some mathematicians, indeed, go so far as to regard a theorem as only plausible if the proof has involved the use of transfinite numbers.

It has long seemed to the authors that a proof of this fundamental and important result which used only those concepts which occur in its enunciation was a real desideratum, and recently attention was called at a meeting of this Society to the need in question. The present paper contains the complete solution of the problem here proposed.

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* Dini, *op. cit.*, German translation by Lüroth and Schepp, pp. 254–5.

† K. Weierstrass, *Abhandlungen aus der Functionenlehre,*" pp. 97 *seq.* See also Hobson's *Theory of Functions of a Real Variable*, p. 625.

Other analogous results are known, but all of them involve in their very statement the concept of generalised integration. Thus they do not properly form part of the matter in hand. It may be noted, however, that, by means of the main result with regard to the existence of a differential coefficient, all the links in the chain of demonstration of these analogous results of Lebesgue's may be forged without the use of Cantor's numbers* and without, we may add, using any definition of generalised integration other than that based on monotone sequences.†

It will be noticed that we begin by proving a lemma analogous to the Heine-Borel theorem, but differing from it in two important respects: (*a*) the intervals obtained are non-overlapping, (*b*) certain of the points are left uncovered. This lemma, at least in the case when the set S is the whole interval, might be deduced from Lebesgue's lemma concerning his chains of intervals,‡ the proof of which involves Cantor's numbers. Moreover, conversely, Lebesgue's lemma is virtually contained in ours, though the application of it, without Cantor induction, would be difficult. We then use our lemma to prove the theorem concerning the existence of a differential coefficient in the case in which the derivates are bounded. Finally we deduce the more general from the special case by reasoning of an entirely different character, and involving still simpler considerations.§ The paper terminates with two appendices, the first on the connection between Lebesgue's lemma and ours, and the second on the fundamental theorem of integration.

It may be added that the results of the paper, "On the Derivates of Non-differentiable Functions,"‖ in the proof of which only the special case of Lebesgue's theorem was assumed, come, thanks to the present paper, to have a truth independent of the concepts of generalised integration and of Cantor's numbers; in other words, we can now assert that it has been demonstrated without the use of these concepts that there is necessarily a distinction of right and left in the values of the derivates of a nowhere differentiable function at a set of points which is everywhere dense and is of the first category, that the upper and lower

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* There is one possible exception as to this statement, with which the authors are at present occupied. The reader is referred to a paper, by one of them, to be published in the October number of the *Quarterly Journal of Mathematics*, on "Functions of Bounded Variation," where a detailed account of the theorems in question is given.

† W. H. Young, "A New Method in the Theory of Integration," 1910, *Proc. London Math. Soc.*

‡ "Functions of Bounded Variation," p. 73.

§ This portion of the proof is repeated here for the convenience of readers from the paper quoted above.

‖ *Messenger of Mathmatics*, 1908, p. 65.

bounds of the derivates at the points of this set are respectively $+\infty$ and $-\infty$, that at the remaining points of the interval both the upper derivates are $+\infty$ and both the lower derivates are $-\infty$, exception being made of at most another set of the first category, and that, finally, the function itself cannot in any interval have bounded variation.

2. LEMMA.—*If with each point of a closed set S as left-hand end-point we have given an interval, or several intervals, we can find a finite number of these, nowhere overlapping, and such that the content of the sub-set of S external to the chosen intervals is less than any pre-assigned positive quantity e.*

Let n be any chosen integer, and

$$e_1 > e_2 > e_3 > \dots$$

a monotone descending sequence of positive quantities, having zero as limit.

Let us elongate each of the given intervals $(x,\ x+h)$ by one n-th of its length, or less, in such a way that, instead of the single interval $(x,\ x+h)$, we have the set of intervals $(y,\ x+h)$, where

$$x - \frac{h}{n} \leqslant y \leqslant x.$$

Let a be the left-hand end-point of the closed set S. Then the point a determines the sub-set of the original intervals corresponding to elongated intervals having a as left-hand end-point. Therefore, also, a determines e_{r_1}, the *first* of the quantities e_i of the above e-sequence which is not greater than all the original intervals of that sub-set.

Let us choose as the first of the intervals sought an original interval d_1 of the above mentioned sub-set, such that

$$d_1 \geqslant e_{r_1};$$

the interval d_1', got by elongating* d_1 to the point a, is then one of the intervals we obtained at the commencement by elongation.

The points of S which do not lie to the left of the right-hand end-point of d_1 form a closed set, which we now proceed to treat in the same way as before. We thus obtain an integer r_2, and choose an interval d_2 of the original set, such that

$$d_2 \geqslant e_{r_2},$$

and a corresponding interval d_2' of the elongated set, which will abut with

* d_1' may, of course, be identical with d_1.

d_1' if the right-hand end-point of d_1' is a point of the closed set S, and otherwise will be such that between d_1' and d_2' there is no point of that closed set.

Continuing the process here indicated, we obtain a succession of the original intervals

$$d_1,\ d_2,\ \ldots,$$

and a corresponding succession of the elongated intervals

$$d_1',\ d_2',\ \ldots,$$

these latter leaving no point of S between them uncovered.

We shall now shew that, after a finite number of such intervals have been chosen, the right-hand end-point of the last interval is within a distance $\frac{1}{2}e$ of the right-hand end-point b of the closed set S.

For, if not, the countably infinite set of non-overlapping intervals d_i determines a single limiting point c to the right of all of them, but to the left of b. Since each d_i contains a point of the set S, c is a point of S, since that set is closed. Therefore there is an interval $(c,\ c+h_c)$ of the original set, and this determines the *first* integer r_c such that

$$e_{r_c} \leqslant h_c.$$

Also, since c is a limiting point of the intervals d_i, there is one of them, say d_k, inside the interval $(c-1/n.h_c,\ c)$, so that the sub-set of the original intervals from which d_k was chosen contains the interval $(c,\ c+h_c)$, which is $\geqslant e_{r_c}$. But r_k is the *smallest* integer for which any interval of this sub-set $\geqslant e_{r_k}$; therefore r_k is not greater than r_c. Hence

$$e_{r_c} \leqslant e_{r_k} \leqslant d_k,$$

since d_k was so chosen.

By precisely the same argument,

$$e_{r_c} \leqslant d_{k+1};$$

and so on. But this is manifestly absurd, since all the intervals $d_k,\ d_{k+1},\ \ldots$ are internal to the interval $(c-1/n.h_c,\ c)$. Thus the contingency here contemplated is impossible.

This proves the statement already made, that after a finite number, say m, of intervals have been chosen, the right-hand end-point of the last interval is within a distance $\frac{1}{2}e$ of b. Thus the elongated intervals $d_1',\ d_2',\ \ldots,\ d_m'$ cover up the whole set S to within a distance $\frac{1}{2}e$ of b. Also, by the principle of elongation, the fresh gaps left when we reduce the intervals d_i' again to d_i (in doing which we may uncover some more of the points of S), have a sum less than $(d_1+d_2+\ldots+d_m)/n$, *a fortiori*

less than G/n, where G is the content of the whole set of original intervals. Thus our chosen intervals $d_1, d_2, \ldots, d_m$ cover up the whole of S excepting a sub-set of content less than $\frac{1}{2}e+G/n$, which is less than e, provided n was chosen greater than $2G/e$. This, therefore, proves the lemma.

3. THEOREM.—*If $f(x)$ is a monotone increasing function whose derivates are bounded, it has a differential coefficient, except at a set of content zero.*

Let G_k denote the set of points at which

$$f^+(x)-f_+(x) > k. \tag{1}$$

Then, if k assumes in succession the values $A, \frac{1}{2}A, \frac{1}{4}A, \ldots$, where A is the upper bound of the derivates of f, either there is some value of k for which G_k has a positive content or else every set G_k has content zero, in which case the outer limiting set of the sets G_k has content zero, that is, the set of all the points at which the right-hand derivates do not coincide has content zero.

Suppose then, if possible, that there is one of these values of k such that G_k has positive content. Now the set G_k is itself the sum of the finite number of sets $H_{y,k}$ at which, besides (1), we have

$$\tfrac{1}{2}(y-1)k \leqslant f_+(x) < \tfrac{1}{2}yk, \tag{2}$$

y denoting any positive integer up to that for which $\frac{1}{2}(y-1)k = A$. Hence at least one of these sets must have positive content. Suppose it is $H_{y,k}$, then there must be a closed sub-set of $H_{y,k}$ of positive content S.

Let e be a sufficiently small positive quantity to satisfy the inequality

$$\tfrac{1}{2}kS > e\{(y+1)k+2A\}, \tag{3}$$

and let us divide the fundamental segment into equal compartments, so small that the sum of those which contain the points of the closed set S is less* than $S+e$.

Now let x be any point of the closed set S, and therefore an internal point, or a left-hand end-point of one of the compartments above mentioned. Then, by (1) and (2),

$$f_+ < \tfrac{1}{2}yk, \tag{4}$$

and

$$\tfrac{1}{2}(y+1)k < f^+(x). \tag{5}$$

* This was the basis of Hankel's definition of the content of a closed set. Cp. Young's *Theory of Sets of Points*, p. 86.

Therefore we can find two intervals $(x, x+h_1)$ and $(x, x+h_2)$ each inside the compartment determined above by the point x, and such that

$$f(x+h_1)-f(x) < \tfrac{1}{2}ykh_1 \tag{6}$$

and

$$\tfrac{1}{2}(y+1)kh_2 < f(x+h_2)-f(x). \tag{7}$$

By the lemma of § 3, we can choose out a finite number of non-overlapping intervals in each case from the sets (6) and (7), so that the sum of the lengths of the chosen intervals is in each case greater than $S-e$, and the content of the points of S not internal to them is less than e. In each case the set so chosen will leave over a finite set of intervals complementarily filling up the compartments containing the set S. Since the content of those compartments is less than $S+e$, it follows that the content of these complementary intervals is less than $2e$, and therefore the increment of the function $f(x)$ over them is less than $2eA$, since the derivates, and therefore the incrementary ratio, of f are all $\leqslant A$.

Hence, if P denote the increment of f over the remaining compartments which contain no points of S as internal points, and P_1 and P_2 respectively the increments over the chosen intervals from the sets (6) and (7), we have, since the sum of the increments from end to end of the fundamental segment is the same

$$P+P_2 < P+P_1+2eA.$$

But, by (6) and (7),

$$\tfrac{1}{2}(y+1)\,k(S-e) < P_2, \quad \text{and} \quad P_1 < \tfrac{1}{2}yk(S+e),$$

since the content of the chosen intervals is greater than $S-e$ and less than that of the compartments in which they lie, that is, less than $S+e$.

Hence

$$\tfrac{1}{2}(y+1)\,k(S-e) < \tfrac{1}{2}yk(S+e)+2eA,$$

whence, *a fortiori*,

$$\tfrac{1}{2}kS < e[(y+1)k+2A],$$

which is in contradiction to (3).

Thus it is impossible that any one of the sets G_k can have positive content, so that the set of points at which the upper and lower right-hand derivates are different has content zero.

Similarly, the set of points at which the upper and lower left-hand derivates are different has content zero ; and again, similarly, the points at which the upper derivate on one side differs from the upper derivate on the other side has content zero. Hence, finally, the points at which there is not a differential coefficient form a set of zero content, which proves the theorem.

COR.—*If $f(x)$ is any function having bounded derivates, it has a differential coefficient except at a set of content zero.*

For, if the derivates are all greater than $-A$, the derivates of $f(x)+Ax$ are all positive and bounded, so that by the theorem this function has a differential coefficient except at a set of content zero, whence the same is, of course, true for $f(x)$.

4. THEOREM.—*Every monotone function possesses a finite differential coefficient, except at a set of content zero.*

It is evidently sufficient to prove the theorem when all the derivates of the function are greater than some chosen positive quantity A. For, if the theorem is true for such a function, it is true for any monotone increasing function, since we change such a function into one whose derivates are greater than A by adding $2Ax$, and a monotone decreasing function becomes monotone increasing by changing its sign.

Assuming, then, all the derivates to be greater than A, the function $y = F(x)$ is monotone increasing, without any tracts of invariability, and may be continuous or discontinuous. The inverse function $x = G(y)$ is therefore a continuous monotone increasing function of y, without or with tracts of invariability, and its derivates lie between $1/A$ and zero, the former value excluded. Thus, by the theorem just proved, $G(y)$ possesses a differential coefficient except at a set of content zero of values of y.

Now the incrementary ratio of the function $G(y)$ has the same upper bound as the derivates, and is therefore less than $1/A$. Describe round the exceptional set of values of y a countable set of non-overlapping intervals of content I as small as we please. Let d be the length of one of these intervals. Then the values of x corresponding to the values of y at the ends of this interval necessarily differ by less than d/A, since the incrementary ratio is less than $1/A$. Hence the values of x corresponding to the exceptional values of y are contained in a set of intervals whose sum is less than I/A, so that the content of the set of values of x corresponding to the exceptional set of values of y is less than I/A. Since I may be as small as we please, this proves that the set of values of x in question is also of zero content. Thus we have shewn that $G(y)$ possesses a differential coefficient except for values of y corresponding to a set of values of x of zero content. Hence $F(x)$ has a differential coefficient, finite or infinite, except for a set of values of x of zero content.

Again, where the differential coefficient of $F(x)$ is infinite it is $+\infty$. Hence the points where this occurs form a set of zero content. For, if

the content were I, we could surround the set in question by non-overlapping intervals, finite or countably infinite in number, such that the incrementary ratio of $F(x)$ over each such interval was greater than B, where B is as large a positive quantity as we please, and therefore the sum of the corresponding increments of $F(x)$ would be greater than BI. Hence the net increment of the function over the whole interval, the function $F(x)$ being monotone increasing, would be greater than BI, which is only possible if I is zero, since B is as great as we please, and $F(x)$ is bounded.

Thus $F(x)$ possesses a differential coefficient except at a set of content zero, and this differential coefficient is itself only infinite at a set of content zero, which proves the theorem.

Cor.—*Every function of bounded variation possesses a finite differential coefficient except at a set of content zero.*

For such a function is the difference of two monotone increasing functions.

Appendix I.

On the Virtual Identity of a Special Case of our Lemma with that of Lebesgue.

5. From our lemma we have the following corollary:—

Cor.—*A countable non-overlapping set of the given intervals can be found such that every point of the closed set S is an internal or end-point of the selected intervals.*

For, let the process described in the proof of the lemma be performed in such a way that the extreme right-hand complementary interval is of length less than G/n, and let us proceed a stage further by repeating this process in each of the complementary intervals of the selected $d_1, d_2, \ldots, d_m$, and then again repeating the process, and so on. Then, either after a finite number of stages there is no point of S left external to the intervals selected, in which case the statement is verified, or else we go on *ad infinitum* without ever arriving at such a final stage. With this latter hypothesis we can also shew, by a *reductio ad absurdum*, that the statement is true.

Suppose, in fact, if possible, that there is a point c of the set S which is at every stage external to the selected intervals. Then there is an interval $(c, c+h_c)$ of the original set, and this determines the first integer r_c such that

$$e_{r_c} \leqslant h_c.$$

At the end of the first stage the extreme right-hand gap, as well as each of the gaps introduced by reducing the elongated intervals $d_1', d_2', \ldots, d_m'$ down again to $d_1, d_2, \ldots, d_m$, is less than G/n, where G is the content of the original set of intervals. Therefore at the second stage the gaps introduced in the same way by cutting off the elongated parts will be less than G/n^2, and so on. Thus we can determine a stage from which onwards each of the gaps introduced in this way is less than h_c/n.

Now, since c is a point of S, but not internal to the selected intervals at any stage, it is at each stage internal to one of these gaps. Hence we see that, at the stage at which we have just arrived, the right-hand end-point P of one of the intervals selected lies inside the interval $(c-h_c/n, c)$. Hence, at the next stage, the nearest point of S used as starting point will be either the point c or lie between c and P inclusive, so that it is certainly a point of the interval $(c-h_c/n, c)$, whence it follows that the interval $(c, c+h_c)$ belongs to the subset from which the corresponding choice of, say, the interval d_k is made. Hence

$$e_{r_c} \leqslant d_k,$$

Similarly, at the next and each subsequent stage, there will be another interval selected of length $\geqslant e_{r_c}$, which is impossible if the total content G of the given intervals is finite, as is assumed to be the case. Thus there can be no point of S which is not internal to the selected intervals or an end-point of them.

When the set S consists of all the points of a closed interval* (a, b), the argument shews that no point c of the closed set S can be a right-hand end-point of one of the selected intervals without being a left-hand end-point of another. Hence this special case is identical with that contemplated by Lebesgue, and the intervals form a Lebesgue chain reaching from a to b.

On the other hand, the special case of our lemma follows immediately if we assume Lebesgue's lemma, viz.,

If with each point x of the interval (a, b) as left-hand end-point we have an interval or several intervals, we can find a chain of the intervals overlapping nowhere and such that every point of the interval (a, b), except b, is an internal point or a left-hand end-point of an interval of the chain.†

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* It is obviously immaterial whether the point b is included in the interval or not.

† This lemma is constantly used by Lebesgue, though never formally enunciated by him. Cp., for example, *Leçons sur l'Intégration*, p. 62. The statement given above is taken from "Functions of Bounded Variation."

For in this case, by Cantor's theorem, the intervals are countable, being non-overlapping, and their lengths form a convergent series whose sum is (a, b), since there is no point of (a, b) external to them. Hence we can choose out a finite number of the intervals so that the sum of their lengths is greater than $(b-a-e)$. The sum of the complementary intervals is therefore less than e, which proves our lemma.

Appendix II.

On the Fundamental Theorem of Lebesgue Integration for a Function with Finite Derivates.

6. In this Appendix we propose to justify the statement, made in the introduction, as regards the fundamental theorem of integration in the case, first, of a function with bounded derivates, and, secondly, under somewhat more general conditions.

Theorem.—*A function with bounded derivates is the integral of any one of its derivates.*

Consider the oscillating succession of functions

$$\frac{f(x+h)-f(x)}{h}.$$

Since this incrementary ratio for all values of x and h has the same upper and lower bounds as the derivates of f, the succession is bounded both above and below, and is therefore* semi-integrable both above and below, that is,

$$\int_a^x f_+(x)\,dx \leqslant \operatorname*{Llt}_{h=0} \int_a^x \frac{f(x+h)-f(x)}{h}\,dx \leqslant \int_a^x f^+(x)\,dx,$$

taking h to be positive.

But the succession of integrals converges to $f(x)-f(a)$, and the upper and lower right-hand derivates f^+ and f_+ are equal, except at a set of content zero, by the corollary in § 3. Hence

$$f(x)-f(a) = \int_a^x f_+(x)\,dx = \int_a^x f^+(x)\,dx,$$

and, similarly, the equality with the integrals of the left-hand derivates, taking h negative, follows.

Theorem.—*A function whose derivates are bounded, except in the*

* W. H. Young, "Term-by-term Integration of Oscillating Series," 1909, *Proc. London Math. Soc.*, Ser. 2, Vol. 8, p. 111. A slight and obvious correction is needed in lines 14 and 15, and is given in the paper on "Semi-integrals and Oscillating Successions of Functions," by one of us, recently presented to the Society. The expression "semi-integrable" is introduced in this latter paper.

neighbourhood of a countable set of points, is the integral of any one of its derivates provided it has bounded variation.

It follows, indeed, also from the theory of oscillating successions that the above succession is semi-integrable both above and below.* For $f(x)-f(a)$ is continuous, and the points in which the peak and chasm functions of the oscillating succession are infinite are countable.

In fact, since the upper bound of the derivates in any interval is the same as that of the incrementary ratio,† the points in the neighbourhood of which the derivates are unbounded above are the same as those in the neighbourhood of which the incrementary ratio $[f(x+h)-f(x)]/h$, $(0 < h)$, regarded as a function of (x, h), is unbounded above. These points evidently form a closed set, which is, by hypothesis, countable.‡

But, $f(x)$ having bounded variation, its derivates agree except at a set of content zero and therefore, being summable,§ have the same integral, which must therefore be equal to $f(x)-f(a)$. This proves the theorem.‖

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* *Id.*, pp. 113, 114.

† So that the peak-function of the succession of incrementary ratios is the upper limiting function of the derivates.

‡ [*Added March 15th*, 1911.—It follows from § 3 (Cor.), that *a function whose derivates are bounded except at a closed set of content zero has a finite differential coefficient except at a set of content zero.* Hence, as at the end of § 6, we see that *a function whose right (left) hand derivates are summable and are bounded except at the points of a closed countable set is the integral of any one of its derivates.* For further results obtained without Cantor induction, see my "Note on the Fundamental Theorem of Integration" (1910), *Proc. Camb. Phil. Soc.*, Vol. XVI, pp. 35–38.]

§ This follows as a corollary to the theorem of § 15 of "Semi-Integrals," these *Proceedings*, p. 300, since by that theorem the lower (upper) derivate of a monotone ascending (descending) function is summable.

‖ This page has been modified during its passage through the press.

THE DOUBLE SIX

By G. T. Bennett.

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1. The following paper consists of a series of notes, partly analytical and partly synthetic, dealing with the double six of lines and the geometry immediately associated with that configuration. The work of previous authors makes it unnecessary to offer any strictly consecutive treatment of the form, and allows the present paper, without disadvantage, to be of a miscellaneous and supplementary character.

The double six has received attention both as a grouping of lines of the cubic surface and as an independent configuration of line geometry, but has been less noticed in connection with the theory of the quadric surface. This last view will predominate in what follows, and will be discussed first.

2. *Lines Mutually Conjugate with respect to a Quadric.*

By the conjugacy of two lines a_1, a_2 is meant the one-fold relation subsisting when a_1 meets b_2 the polar line of a_2, and a_2 meets b_1 the polar line of a_1. The lines b_1, b_2 are at the same time also conjugate. The point $a_1 b_2$ is conjugate to all points on a_2, and the point $a_2 b_1$ is conjugate to all points on a_1. An alternative and equivalent definition of the conjugacy of a_1 and a_2 is to state that conjugate points on the two lines, instead of forming two projective ranges, give a factorising homography for which all points of either line correspond to a single point of the other line. The same may be said (*mutatis mutandis*) of planes through the lines.

It is natural to enquire in respect of lines, as of points and planes, how many may be mutually conjugate. In the case of points or planes a set of not more than four may be mutually conjugate, and the configuration, in each case a tetrahedron, has no singularity as a configuration. In the case of lines the result is very different. When four lines a_1, a_2, a_3, a_4 have been taken mutually conjugate, the addition of a further line conjugate to these four is a determinate quadratic problem ; for the additional line must be either of those two lines, a_5 or a_6, which meet b_1, b_2, b_3, b_4.

The fundamental peculiarity which then presents itself is that the lines a_5 and a_6 are themselves conjugate. The theorem, namely, which awaits proof is this : that *if six lines, taken in pairs in all fifteen ways, are conjugate in every case but one, then the pair omitted are also conjugate lines.* An equivalent statement, more concise but less symmetrical, is to say that *if four lines are mutually conjugate, then the two lines which meet them are also conjugate.* Or, as another alternative, *if five lines are mutually conjugate, they have a common transversal.* The theorem will be dealt with in different ways in §§ 4, 6. The truth of it involves immediately the appearance of the double six of lines $a_1 a_2 a_3 a_4 a_5 a_6$ and $b_1 b_2 b_3 b_4 b_5 b_6$, each line meeting all lines of the opposite system except its companion polar line.

3. *If three generators of a quadric S' are mutually conjugate with respect to a quadric S, then the property is poristic, and there exist an infinity of such triads of generators.* The generators may belong to either system.

The theorem is given by Schur (*Mathematische Annalen*, Bd. 18, 1881). It leads (§ 4) to a simple proof of the theorem of § 2. It is here exhibited in a form which shows an intimate relation of the theorem to a familiar porism of plane geometry.

The quadrics may be written in the form

$$S \equiv x^2 + y^2 + z^2 + t^2 = 0,$$

$$S' \equiv px^2 + qy^2 + rz^2 + pqrt^2 = 0.$$

Three lines

$$a_1 (l_1, m_1, n_1, pl_1, qm_1, rn_1),$$

$$a_2 (l_2, m_2, n_2, pl_2, qm_2, rn_2),$$

$$a_3 (l_3, m_3, n_3, pl_3, qm_3, rn_3),$$

are generators of S', if

$$pl_1^2 + qm_1^2 + rn_1^2 = 0,$$

$$pl_2^2 + qm_2^2 + rn_2^2 = 0,$$

$$pl_3^2 + qm_3^2 + rn_3^2 = 0,$$

and are mutually conjugate with respect to S if

$$(1+p^2)\, l_2 l_3 + (1+q^2)\, m_2 m_3 + (1+r^2)\, n_2 n_3 = 0,$$

$$(1+p^2)\, l_3 l_1 + (1+q^2)\, m_3 m_1 + (1+r^2)\, n_3 n_1 = 0,$$

$$(1+p^2)\, l_1 l_2 + (1+q^2)\, m_1 m_2 + (1+r^2)\, n_1 n_2 = 0.$$

It follows that the lines a_1, a_2, a_3 meet the plane $t = 0$ in the vertices of

a triangle which is inscribed in the conic

$$t = 0, \quad px^2+qy^2+rz^2 = 0,$$

and self-polar with respect to the conic

$$t = 0, \quad (1+p^2)\,x^2+(1+q^2)\,y^2+(1+r^2)\,z^2 = 0.$$

The relation is poristic, and the invariant condition is

$$\frac{p}{1+p^2}+\frac{q}{1+q^2}+\frac{r}{1+r^2}=0. \tag{1}$$

It follows that the existence of a single triad of generators of a quadric S', mutually conjugate with respect to a quadric S, implies the existence of an infinity of such triads. Further, since a reversal of sign of p, q, r leaves the condition (1) unaltered, it follows that the second system of generators of S' also contains a single infinity of such triads.

The equation (1) may be written in the form

$$(p+q+r+pqr)\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}+\frac{1}{pqr}\right)=4,$$

or, in terms of the invariants of the quadrics S, S',

$$\Theta\Theta'-4\Delta\Delta'=0. \tag{2}$$

The symmetry of this result shows that the relation between the quadrics is reciprocal; and that therefore triads of generators of either quadric, and of either system, are mutually conjugate with respect to the other quadric.

4. *Schur's Theorem and the Double Six.*

The connection between the theorem of Schur, the theorems of § 2, and the existence of the double six are seen easily as follows. Four lines a_1, a_2, a_3, a_4 are taken mutually conjugate with respect to the quadric S. The lines a_1, a_2, a_3, as generators, determine a quadric S'. Schur's theorem is true for these two quadrics. Taking b_4 the polar of a_4 with regard to S, it meets $a_1a_2a_3$, and therefore belongs to the second system of generators of S'. With it are associated two other generators of the same system, say b_5 and b_6, the three forming a triad mutually conjugate with respect to S. The two lines b_5 and b_6 which meet the four mutually conjugate lines a_1, a_2, a_3, a_4 are therefore themselves conjugate. And if a_5, a_6 are taken, the polars of b_5, b_6, the conjugacy of a_5 and a_6 appears as a consequence of the conjugacy of all other pairs of the six lines a_1, a_2, a_3, a_4, a_5, a_6. These six lines and their polars b_1, b_2, b_3, b_4, b_5, b_6 constitute the double six. The lines $b_1b_2b_3$ and $a_4a_5a_6$ are generators of a third

quadric S'', which is the polar reciprocal of S' with regard to S. The quadrics S', S'' have been treated by Burnside (*Proc. Camb. Phil. Soc.*, Vol. 14, 1908), whose results may be compared with the foregoing.

5. *A Symmetrical Type of Double Six.*

If the quadrics S and S', and therefore also S'', are ruled hyperboloids of revolution with a common axis, a special form of double six arises which may be described simply in terms of a right hexagonal prism. The top and bottom (as they may conveniently be called) are equal hexagons; each having its angles all equal and its sides alternately equal. The six vertical faces are alternately equal rectangles. The diagonals of these faces form the double six. The diagonals a_1, a_2, a_3, a_4, a_5, a_6 are taken consecutively, one in each face, so that they have all the same screw sense in reference to the axis of the prism, and are consequently non-intersecting lines. The remaining diagonals, taken in the same order, are then $b_4 b_5 b_6$, $b_1 b_2 b_3$. The figure is symmetrical about each of the three axial planes perpendicular to the pairs of parallel faces, these three planes being mutually inclined at angles of 60°. It might be referred to as the "kaleidoscopic" double six. It exhibits many of the essential properties of the general double six in a concise form, but it is a degenerate type; for the quadrics S', S'' intersect in the circumcircles of the hexagonal ends of the prism (two conics that is) instead of the twisted quartic curve of the general case.

6. *Planes touching the Polar Quadric S and passing through a Pair of Lines of the Double Six, taken as Planes of Reference.*

Take any one pair of lines of the double six, say a_6 and b_6. Let the tangent planes to S through b_6 be $x = 0$, $y = 0$, and through a_6 be $z = 0$, $t = 0$. Then, since the lines a_6, b_6 are polar lines, the quadric may, without loss of generality, be written

$$S \equiv xy - zt = 0.$$

The line a_1, conjugate to a_6 and meeting b_6, may be taken as

$$x - l_1 y = 0, \quad z + n_1 y + m_1 t = 0,$$

with coordinates $a_1 (l_1, 1, -n_1, -m_1, l_1 m_1, 0)$.

The line b_1, the polar line of a_1, is then given by equations

$$z - m_1 t = 0, \quad x + l_1 y + n_1 t = 0,$$

with coordinates $b_1(-l_1, 1, 0, m_1, l_1m_1, n_1)$.

Corresponding results, with suitable change of suffix, hold for the other pairs of lines a_2b_2, a_3b_3, a_4b_4, a_5b_5.

The condition of conjugacy of a_1 and a_2 now takes the form

$$(l_1+l_2)(m_1+m_2) = n_1n_2. \tag{3}$$

The existence theorem of the double six demands that the complete set of ten equations of this type should be poristic; namely, that they should be equivalent to only nine independent equations, any one of the equations being true as a consequence of the rest.

7. *Poristic Property of the Ten Equations of Type* $(l_1+l_2)(m_1+m_2) = n_1n_3$.

A direct proof is obtainable as follows. Let the equations be denoted by the symbols $(\overline{12})$, $(\overline{13})$, ..., $(\overline{45})$; and let the last equation $(\overline{45})$ be left to appear as a consequence of the other nine. Consider three other equations

$$(l+l_1)(m+m_1) = n_1n,$$

$$(l+l_2)(m+m_2) = n_2n,$$

$$(l+l_3)(m+m_3) = n_3n,$$

which may be denoted $(\bar{1})$, $(\bar{2})$, $(\bar{3})$. Then, in virtue of equations $(\overline{14})$, $(\overline{24})$, $(\overline{34})$, the equations $(\bar{1})$, $(\bar{2})$, $(\bar{3})$ are satisfied by $l = l_4$, $m = m_4$, $n = n_4$; and in virtue of $(\overline{15})$, $(\overline{25})$, $(\overline{35})$ they are satisfied by $l = l_5$, $m = m_5$, $n = n_5$. Further, the equations $(\bar{1})$, $(\bar{2})$, $(\bar{3})$ give a quadratic eliminant in any one of the three quantities l, m, n; so that the values $l_4m_4n_4$ and $l_5m_5n_5$ are the two solutions. To verify the equation $(\overline{45})$ it is only necessary to find the values of l_4+l_5, m_4+m_5, and n_4n_5 from the three quadratics.

The quadratic in l is

$$\Sigma(m_2-m_3)\,n_1\,(l+l_2)(l+l_3) = 0,$$

where the summation symbol implies circulation of suffixes 1, 2, 3. Hence

$$\begin{aligned}(l_4+l_5)\,\Sigma(m_2-m_3)\,n_1 &= -\Sigma(l_2+l_3)(m_2-m_3)\,n_1 \\ &= +\Pi n_1(m_2-m_3)/(m_2+m_3),\end{aligned}$$

on using equations $(\overline{23})$, $(\overline{31})$, $(\overline{12})$. [The product symbol implies circulation of suffixes 1, 2, 3.]

Similarly $(m_4+m_5)\,\Sigma(l_2-l_3)\,n_1 = +\Pi n_1\,(l_2-l_3)/(l_2+l_3)$.

Hence
$$(l_4+l_5)(m_4+m_5) = \frac{\Pi(l_2-l_3)(m_2-m_3)}{\Sigma(l_2-l_3)\,n_1\,.\,\Sigma(m_2-m_3)\,n_1}, \qquad (4)$$

and it remains to be shown that this is the value of the product $n_4 n_5$.

The quadratic in n obtained by eliminating l and m from the equations (1), (2), (3), homographic in l and m, may be written in the form

$$(l_1m_1-n_1n,\ l_1,\ 1)(l_1m_1-n_1n,\ m_1,\ 1)-(l_1m_1-n_1n,\ l_1,\ m_1)(l_1,\ m_1,\ 1) = 0,$$

where each bracket denotes a determinant of the third order by means of its first row, the other rows being affected by suffixes 2 and 3.

The coefficient of n^2 in this equation is $(n_1,\ l_1,\ 1)(n_1,\ m_1,\ 1)$, which is equal to the denominator of the fraction (4).

The term independent of n is

$$(l_1m_1,\ l_1,\ 1)(l_1m_1,\ m_1,\ 1)-(l_1m_1,\ l_1,\ m_1)(l_1,\ m_1,\ 1),$$

which is identically equal to $\Pi(l_2-l_3)(m_2-m_3)$ the numerator of the same fraction.

Hence
$$(l_4+l_5)(m_4+m_5) = n_4n_5,$$

and the poristic property is established.

It has seemed worth while to give the above investigation on account of the simplicity with which it suggests itself; but the method next to be shown is more concise as a mode of proof, and leads to results capable of geometric interpretation.

8. *Cross Ratios.*

Taking the equations $(\overline{23})$ and $(\overline{14})$, $(\overline{31})$ and $(\overline{24})$, $(\overline{12})$ and (34), and multiplying them in pairs, we get

$$\begin{aligned}&(l_2+l_3)(l_1+l_4)(m_2+m_3)(m_1+m_4)\\ &= (l_3+l_1)(l_2+l_4)(m_3+m_1)(m_2+m_4)\\ &= (l_1+l_2)(l_3+l_4)(m_1+m_2)(m_3+m_4).\end{aligned} \qquad (5)$$

Now consider the cross-ratio equations

$$\left.\begin{aligned}(l_1\ l_2\ l_3\ l_4) &= (m_1^2 m_2^2 m_3^2 m_4^2)\\ (m_1m_2m_3m_4) &= (l_1^2\ l_2^2\ l_3^2\ l_4^2)\end{aligned}\right\}. \qquad (6)$$

On multiplying these two there is obtained the equality of two of the

three expressions (5); and, since the symbols of the cross ratios may be permuted, any two of the expressions (5) are equal in consequence of (6). The equations (6) are equivalent to the equations (5).

Similarly, from equations involving suffixes 1, 2, 3, 5, we obtain

$$\left.\begin{aligned}(l_1\ l_2\ l_3\ l_5) &= (m_1^2 m_2^2 m_3^2 m_5^2)\\ (m_1 m_2 m_3 m_5) &= (l_1^2\ l_2^2\ l_3^2\ l_5^2)\end{aligned}\right\}. \tag{7}$$

The only equation of the ten not used is $(\overline{45})$. Now, from (6) and (7), it follows that

$$(l_1\ l_2\ l_4\ l_5) = (m_1^2 m_2^2 m_4^2 m_5^2),$$
$$(m_1 m_2 m_4 m_5) = (l_1^2\ l_2^2\ l_4^2\ l_5^2),$$

and hence, by multiplication of these last, it follows that the equation $(\overline{45})$ is a consequence of the other nine. The cross-ratios thus afford an immediate proof of the poristic property.

9. *The Homographic Relations* $(l^2) = (m)$ *and* $(l) = (m^2)$.

The preceding results may be put in the form

$$m = \lambda\,(l^2-\alpha')/(l^2-\alpha), \tag{8}$$

$$m^2 = \mu\,(l-\beta')/(l-\beta), \tag{9}$$

where, in each equation, the identical suffixes of l and m are suppressed. The corresponding value of n may also be expressed in terms of l. For, from these equations,

$$m_1^2-m_2^2 = \mu\,(\beta-\beta')(l_1-l_2)/(l_1-\beta)(l_2-\beta),$$

$$m_1-m_2 = \lambda\,(\alpha-\alpha')(l_1^2-l_2^2)/(l_1^2-\alpha)(l_2^2-\alpha),$$

and hence
$$(l_1+l_2)(m_1+m_2) = \frac{\mu\,(\beta-\beta')}{\lambda\,(\alpha-\alpha')}\left(\frac{l_1^2-\alpha}{l_1-\beta}\right)\left(\frac{l_2^2-\alpha}{l_2-\beta}\right).$$

Consequently
$$n = \kappa\,(l^2-\alpha)/(l-\beta), \tag{10}$$

where
$$\kappa^2 = \mu\,(\beta-\beta')/\lambda(\alpha-\alpha'). \tag{11}$$

As the set of ten equations are equivalent to nine it should be possible to express the m's and n's, save for a scalar constant, say λ, in terms of the l's. It remains to point out, therefore, how the quantities $\alpha, \alpha', \beta, \beta', \mu$ are expressible in terms of the five values of l, namely l_1, l_2, l_3, l_4, l_5, and λ.

Let the equation of which the five l's are roots be written

$$l(l^4+2ql^2+s)+(pl^4+2rl^2+t) = 0, \tag{12}$$

so that $$p = -\Sigma l_1, \quad 2q = \Sigma l_1 l_2, \quad \ldots .$$

An equation equivalent to it arises on eliminating m from equations (8), (9); namely,

$$\lambda^2(l^2-\alpha')^2(l-\beta)-\mu(l^2-\alpha)^2(l-\beta') = 0. \tag{13}$$

If this latter is written

$$l[\lambda^2(l^2-\alpha')^2-\mu(l^2-\alpha)^2]-[\beta\lambda^2(l^2-\alpha')^2-\beta'\mu(l^2-\alpha)^2] = 0, \tag{14}$$

it is seen that α, α' are roots of the Jacobian of the quadratics in l^2 which occur in the brackets of (12); namely, α, α' are roots of

$$(pq-r)\alpha^2+(ps-t)\alpha+(rs-qt) = 0. \tag{15}$$

This quadratic is also the eliminant of the equations

$$\left.\begin{aligned} \alpha\beta+p\alpha+q\beta+r &= 0 \\ q\alpha\beta+r\alpha+s\beta+t &= 0 \end{aligned}\right\}, \tag{16}$$

either of which gives the unique value of β (or β') corresponding to α (or α').

The quantities α, α', β, β' are thus explicitly expressible in terms of the l's. The remaining quantity μ is obtainable at once, by comparison of (12) and (13) in either of the forms

$$\left.\begin{aligned} \mu &= \lambda^2(\beta+p)/(\beta'+p) \\ \mu &= \lambda^2(\alpha'+q)/(\alpha+q) \end{aligned}\right\}, \tag{17}$$

and this completes the expression of α, α', β, β', μ, and so of the m's and n's, in terms of l_1, l_2, l_3, l_4, l_5 and λ.

The lines of the double six are thus presented as five pairs of lines with coordinates symmetrically and explicitly expressed in terms of six independent parameters, together with the completing pair $x = 0$, $y = 0$, and $z = 0$, $t = 0$.

10. *The Cubic Surface containing the Double Six.*

The cubic on which the twelve lines of the double six lie is easily found to be

$$\kappa[(x^2-\alpha y^2)z-\lambda(x^2-\alpha' y^2)t]+[(x-\beta y)z^2-\mu(x-\beta' y)t^2] = 0. \tag{18}$$

Cut by the plane $x-ly=0$, the cubic has a section consisting of the line $x=0$, $y=0$, *i.e.*, b_6, and a conic given by

$$x-ly=0, \quad \kappa[(l^2-a)z-\lambda(l^2-a')t]y+[(l-\beta)z^2-\mu(l-\beta')t^2]=0.$$

If l has one of the five values which are roots of the equation (13), say l_1, the first bracket is a factor of the second, and the conic becomes the two straight lines

$$x-l_1y=0, \qquad (z-m_1t)(n_1y+z+m_1t)=0,$$

of which one is the line a_1, and the other the line c_{16} (in the usual notation).

The section of the cubic by the plane $z-m_1t=0$ is found, similarly, to consist of the line $z=0$, $t=0$, *i.e.*, a_6, together with the two lines

$$z-m_1t=0, \qquad (x-l_1y)(x+l_1y+n_1t)=0,$$

namely, b_1 and c_{16}. The line c_{16} is thus the line common to the planes a_1b_6 and b_1a_6; and similarly for c_{26}, c_{36}, c_{46}, c_{56}.

The remaining ten lines of the cubic, making the total of twenty-seven, are such as c_{12}, the line common to planes a_1b_2 and a_2b_1. The line c_{12} is given therefore by equations

$$n_1(x+l_2y)+(l_1+l_2)(z+m_1t)=0,$$
$$n_2(x+l_1y)+(l_1+l_2)(z+m_2t)=0.$$

It should be verifiable (i) that this line is a line of the cubic, (ii) that such pairs as c_{12}, c_{34} (with no common suffix) are intersecting lines. These notes are concerned primarily, however, with the theory of the double six itself, and I omit the necessary analysis.

11. *Projective Relations of the Double Six and the remaining Fifteen Lines of the Associated Cubic.*

Any pair of lines a_1, b_1 of the double six being polar lines, it follows that the point a_1b_2 has for polar plane the plane a_2b_1; and similarly the point a_2b_1 has for polar plane the plane a_1b_2. The range of points $a_1(b_2b_3b_4b_5b_6)$ on a_1 is therefore projective with the pencil of planes $b_1(a_2a_3a_4a_5a_6)$; and the pencil of planes $a_1(b_2b_3b_4b_5b_6)$ is projective with the range of points $b_1(a_2a_3a_4a_5a_6)$. The same is true for all other such pairs of ranges and pencils. (19)

A range of collinear points paired with the points where the line is cut by their polar planes with regard to a quadric form an involution. Hence the range $a_1(b_2b_3b_4b_5b_6)$ paired with the points in which a_1 is met

by the planes $b_1(a_2a_3a_4a_5a_6)$ form an involution. The latter points are $a_1(c_{12}c_{13}c_{14}c_{15}c_{16})$. So pairs of points on a_1 given by $a_1(b_2b_3b_4b_5b_6)$ and $a_1(c_{12}c_{13}c_{14}c_{15}c_{16})$ are in involution. Similarly for all other lines. (20)

When the double six is constructed as a configuration of lines, independently of any quadric, and the fifteen additional lines are associated with it, there should appear the theorems (i) that the five pairs of points on each of the lines of the double six are in involution, and (ii) that the twenty-four double points of these twelve involutions all lie on a quadric surface.

If a variable plane is drawn through the line a_1 it is known to touch the cubic at two points belonging to an involution. It is the five triple tangent planes, in particular, which give rise to the five pairs of points of theorem (20).

Similar results may be stated for pencils of planes. Planes $a_1(b_2b_3b_4b_5b_6)$ are in involution with planes projecting from a_1 the points $b_1(a_2a_3a_4a_5a_6)$. The former set meet a_2 in points $a_2(c_{12}b_3b_4b_5b_6)$. Hence planes through a_1 projecting the points $a_2(c_{12}b_3b_4b_5b_6)$ are in involution with the planes projecting the points $b_1(a_2a_3a_4a_5a_6)$. If the first pair of points are omitted the theorem is equivalent to one given by Dixon (*Quarterly Journal*, Vol. 40, 1909), namely, that the points $a_2(b_3b_4b_5b_6)$ are projective with the points $b_1(a_3a_4a_5a_6)$; and that projection from the line a_1 gives a pencil of planes in involution. (21)

Planes through b_2, similarly, projecting the points $a_2(b_1b_3b_4b_5b_6)$ are in involution with planes projecting the points $b_1(c_{12}a_3a_4a_5a_6)$; and of these points the last four pairs are the same as before. They give rise to an involution of planes, therefore, when projected either from b_2 or from a_1.

A geometric interpretation of the homographic relations $(l^2) = (m)$ and $(m^2) = (l)$ of § 9 may now be given. The point a_1b_6 is $(0, 0, -m_1, 1)$; and the point a_6b_1 is $(-l_1, 1, 0, 0)$. The point a_6c_{16}, namely $(l_1, 1, 0, 0)$ is the conjugate, on a_6, of a_6b_1 with regard to S. If an arbitrary point $(\xi, 1, 0, 0)$ is taken on a_6, its harmonic conjugate with regard to the pair of points a_6b_1 and a_6c_{16} is the point $(l_1, -\xi, 0, 0)$. Four such points have a cross-ratio, therefore, given by the cross-ratio of the squares of the corresponding values of l. Hence appears an interpretation of the equation $(l^2) = (m)$; namely, if the harmonic conjugates of an arbitrary point on a_6 are taken with regard to the pairs of points $a_6(b_1b_2b_3b_4b_5)$ and $a_6(c_{16}c_{26}c_{36}c_{46}c_{56})$, then this range of points is projective with the points $b_6(a_1a_2a_3a_4a_5)$. Convenience of statement is secured, with no actual loss of generality, by taking the arbitrary point to be at infinity on a_6. The range derived from the pairs of points $a_6(b_1b_2b_3b_4b_5)$ and $a_6(c_{16}c_{26}c_{36}c_{46}c_{56})$

consists then of the middle points of the segments. The theorem may then (with a change of suffixes) be briefly stated thus :—*The mid-points derived form the involution of pairs given by* $a_1(b_2b_3b_4b_5b_6)$ *and* $a_1(c_{12}c_{13}c_{14}c_{15}c_{16})$ *form a range of points projective with the points* $b_1(a_2a_3a_4a_5a_6)$. (22)

This last theorem may be extended by taking account of all those double sixes, contained in the complete system of twenty-seven lines, which include the line a_1. The involution on a_1 is in each case the same, and so the range of mid-points is the same. But in place of b_1 may occur any one other of the sixteen lines not meeting a_1. On each of these lines occurs a range of points projective with the range of mid-points of a_1. Conversely, on any two lines occur ranges projective with the range of mid-points on any third line which meets neither, and so projective also with each other. If intersecting lines are taken, say a_2 and b_1, and b_2 is associated with them, then the ranges $a_2(b_1b_3b_4b_5b_6)$ and $b_1(a_2c_{13}c_{14}c_{15}c_{16})$ are projective with each other because each is projective with the range of mid-points given by the pairs of points $b_2(a_1a_3a_4a_5a_6)$ and $b_2(c_{21}c_{23}c_{24}c_{25}c_{26})$ on b_2. These projective ranges on a_2 and b_1 have the common point a_2b_1 self-corresponding, and hence the ranges are *in perspective.* That is to say, *the lines joining the points* $a_2(b_3b_4b_5b_6)$ *to the points* $b_1(c_{13}c_{14}c_{15}c_{16})$ *are concurrent.* (23)

It may be observed that the conjugates of the four latter points on b_1, namely $b_1(a_3a_4a_5a_6)$, are those which were formerly (21) seen to be projective with the points $a_2(b_3b_4b_5b_6)$.

There are in all eight lines not meeting either a_2 or b_1 ; being all those, omitting a_2 and b_1 themselves, which meet c_{12}. They are a_1, b_2, c_{34}, c_{56}, c_{35}, c_{46}, c_{36}, c_{45}. Each gives rise, by means of its range of mid-points, to perspective ranges on a_2 and b_1. The set of eight is as follows :—

a_1	gives rise to perspective ranges		$a_2(c_{23}c_{24}c_{25}c_{26})$	and	$b_1(a_3a_4a_5a_6)$,
b_2	,,	,,	$a_2(b_3b_4b_5b_6)$	,,	$b_1(c_{13}c_{14}c_{15}c_{16})$,
c_{34}	,,	,,	$a_2(b_5b_6c_{23}c_{24})$	,,	$b_1(a_6a_5c_{14}c_{13})$,
c_{56}	,,	,,	$a_2(b_3b_4c_{25}c_{26})$	,,	$b_1(a_4a_3c_{16}c_{15})$,
c_{35}	,,	,,	$a_2(b_4b_6c_{23}c_{25})$	,,	$b_1(a_6a_4c_{15}c_{13})$,
c_{46}	,,	,,	$a_2(b_3b_5c_{24}c_{26})$	,,	$b_1(a_5a_3c_{16}c_{14})$,
c_{36}	,,	,,	$a_2(b_4b_5c_{23}c_{26})$	,,	$b_1(a_5a_4c_{16}c_{13})$,
c_{45}	,,	,,	$a_2(b_3b_6c_{24}c_{25})$	,,	$b_1(a_6a_3c_{15}c_{14})$.

The triangle $a_2 b_1 c_{12}$ contains 3×8 such perspectives. The whole may be repeated in turn for each of the forty-five triangles; and hence the total number of such perspective relations in the figure of twenty-seven lines is $3 \times 8 \times 45$ (= 1080).

12. *The Spherical Double Six.*

With no loss of essential generality the polar quadric S of the double six may be taken to be a sphere. The three-dimensional geometry of the double six may then be readily converted into a metrical geometry of the two-dimensional spherical surface. Any line of space is represented by the two points of the sphere through which it passes; the line a_1 being represented by the points $A_1 A_1$ in which a_1 cuts S, and b_1 by the points $B_1 B_1$ in which b_1 cuts S. The lines a_1, b_1 being polar lines, planes through them cut the sphere in two systems of orthogonal coaxal circles; the common points of the two systems being $A_1 A_1$ and $B_1 B_1$. One of these pairs of points is real and the other pair imaginary (the lines of the double six being supposed all real). The pair of real points are the common points of one system of circles and the limiting points of the other system. Either pair of points may be spoken of as the anti-points of the other pair.

If two lines of space meet, then they have a common plane, and the two pairs of points on the sphere are coplanar and therefore concyclic. Thus, since a_1 meets b_2 it follows that $A_1 A_1 B_2 B_2$ are concyclic. And since a_2 meets b_1 the points A_2, A_2, B_1, B_1 are concyclic. These two conditions are equivalent conditions, and either implies the conjugacy of a_1 and a_2, and also of b_1 and b_2. It is convenient to extend the term "conjugacy" to describe the relation of the point pair $A_1 A_1$ to the point pair $A_2 A_2$. The conjugacy may be expressed in terms which relate solely to the two pairs of points themselves. The pair $A_1 A_1$ are conjugate to the pair $A_2 A_2$, if

(α) a circle through $A_1 A_1$ has $A_2 A_2$ as point circles coaxal with it;

or (β) a circle through $A_2 A_2$ has $A_1 A_1$ as point circles coaxal with it;

or (γ) all circles through $A_1 A_1$ cut orthogonally one and the same circle through $A_2 A_2$, and conversely. (Whereas circles through two pairs of arbitrary points, if cutting orthogonally, give rise generally to homographic pairs. Compare § 2 for conjugacy of lines.)

The theorem of the existence of the double six then leads to these theorems of spherical geometry :—(i) if six point-pairs are mutually conjugate in fifteen ways they are conjugate also in the remaining way, or (ii) if five point-pairs are mutually conjugate then there exists a point-pair concyclic with each of the five pairs. Or (iii) if four point-pairs are mutually conjugate then two point-pairs are each concyclic with all four, and these two are themselves conjugate. (24)

The spherical figure also gives a complete representation of the fifteen lines c_{12}, ... For since c_{12} is coplanar with a_1 and b_2 the points C_{12}, C_{12} are concyclic with $A_1A_1B_2B_2$; and since c_{12} is also coplanar with a_2 and b_1 the points C_{12}, C_{12} are concyclic with $A_2A_2B_1B_1$. Hence $C_{12}C_{12}$, points representing the line c_{12}, are the common points of the circles $A_1A_1B_2B_2$ and $A_2A_2B_1B_1$. The twenty-seven lines of the cubic, therefore, are all represented in the figure, and the intersections not already mentioned, those of c_{12} and c_{34}, ..., correspond to further cyclic sets of points $C_{12}C_{12}C_{34}C_{34}$ ....

If a point-pair $C_{12}C_{12}$ are imaginary, then the anti-points D_{12}, D_{12} are real. They represent the line d_{12} joining the points a_1b_2 and a_2b_1, this line being the polar line of c_{12}. In virtue of the intersection of d_{12}, d_{34}, ..., there occur the cyclic set of points $D_{12}D_{12}D_{34}D_{34}$ .... The polar reciprocal of the cubic with regard to the quadric is a surface of the third class having the lines of the double six together with the fifteen lines d_{12}, ... as its twenty-seven lines.

13. *The Plane Inverse of the Spherical Figure.*

The spherical figure may be inverted, from any arbitrary point of the sphere, into a plane figure. In the plane figure there are primarily six point-pairs A_1A_1, ..., and their anti-points B_1B_1, ... Coaxal circles through A_1A_1 have B_1B_1 as limiting points, and conversely. Sets of four such as $A_1A_1B_2B_2$ are concyclic. The "conjugacy" of two pairs A_1A_1 and A_2A_2 is implied by this, or, alternatively, by any one of the conditions (α), (β), (γ) of § 12. A further equivalent condition may be added:— (δ) The products of the lengths of the alternate sides of the quadrilateral $A_1A_2A_1A_2$ are equal. (For the spherical figure this is true of the lengths of the chords.) Theorems (i), (ii), (iii) may be stated in the same terms as in § 12 (24). The point-pairs $C_{12}C_{12}$ and $D_{12}D_{12}$ occur with the same properties as on the sphere.

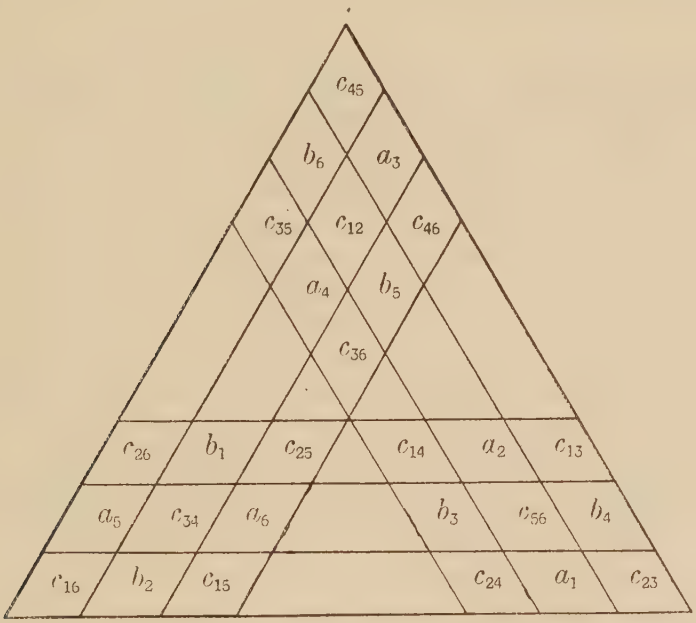
The transformation of the sphere into the plane may, at will, be

regarded as stereographic projection instead of inversion; and this former transformation applies equally to the space-figure associated with the sphere. The lines a_1, b_1, c_{12}, d_{12}, ... become the lines joining the corresponding point-pairs A_1A_1, B_1B_1, $C_{12}C_{12}$, $D_{12}D_{12}$ of the plane figure. If the lines, conveniently, retain the same names, then the points of intersection which occur among the lines of space, a_1b_2, ..., are represented in the plane figure by the points a_1b_2, ... The point a_1b_2, it may be noticed, is the centre of the circle $A_2A_2B_1B_1$, and the point a_2b_1 is the centre of the circle $A_1A_1B_2B_2$. The line c_{12} is the radical axis of the circles; and the line d_{12} is the line of centres.

The homographic and perspective properties of ranges of points given in § 11 for the space figure may be quoted as properties of the plane figure.

14. *A Diagram to show the Intersections and Non-Intersections of the Twenty-Seven Lines of a Cubic Surface.*

To a paper already sufficiently discursive I venture to add the following diagram. It has presented itself in the attempt to devise some topo-



graphical arrangement of twenty-seven symbols in a plane diagram which

should indicate readily, in regard to any two, whether the lines they represent do or do not meet. It is only for convenience of comparison that the symbols entered in the diagram are those which are derived from one particular double six.

The diagram shows nine "strips," in each of which appear six symbols. The overlapping of the strips produces three "corners" in the diagram, each corner containing nine symbols. The rule for intersection is this:— The lines represented by two symbols intersect if the two symbols

(i) are in the same corner and not in the same strip

or (ii) are in different corners and in the same strip.

The rule may be put slightly differently, in a way to comprehend intersections and non-intersections alike. Exhaustive alternatives in regard to two symbols are

(α) they lie in the same corner—or not,

(β) they lie in the same strip—or not,

(γ) the lines they represent meet—or not.

The rule of consistent alternatives is to take the three negatives or only one.

It may be noticed that any one of the strips, for example, gives three non-intersecting lines all met by three other non-intersecting lines. Various groups of lines may be picked out with considerable facility. For three mutually intersecting lines, forming one of the forty-five triangles, the three symbols are found either one in each corner, falling at the vertices of a triangle made by three strips, or all in the same corner, in different rows and columns of the quasi-determinant. The complete set of forty-five triangles are thus accounted for. Or, again, for six lines mutually non-intersecting, the symbols lie either two in each corner, or else three in each of two corners. Each such set of six lines is half of a double six. The twelve lines of a complete double six are given either (1) by the twelve symbols lying in any two parallel strips, or (2) by the symbols lying in three strips forming a triangle, omitting the three common symbols lying at the vertices of the triangle. There are nine of the former and twenty-seven of the latter; and all the thirty-six double sixes are thus accounted for.

[*Note added November 14th*, 1910.—An early proof of the existence of the double six, based on statical considerations, is given by Cayley as a

solution of a Smith's Prize Examination question of the year 1868 (*Collected Papers*, Vol. VIII, pp. 430, 431). The proof, however, though purporting to derive the last intersection from the other twenty-nine, makes use of only nine and neglects twenty, and is therefore necessarily unsound. The statical fallacy occurs at a stage where six forces are obtained as a system in equilibrium and the fact that the forces are separately zero is overlooked.]

THE STABILITY OF ROTATING SHAFTS

By F. B. Pidduck.

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Introduction.

When the time of revolution of the rotating parts of a machine coincides with a natural period of vibration of those parts, there occurs the well known phenomenon of "whirling," and the corresponding speed of revolution is known as the "critical speed." The question of the stability of the steady rotary motion above the critical speed has received some attention from writers on technical mechanics. In treating the important problem of a shaft carrying a disc the method* has been effectively to concentrate the mass in the disc and the elasticity in the shaft. For the purpose of illustration the latter is regarded as a stretched string, but the results are not essentially altered if we take account of the forces called forth by bending and twisting the shaft in the way usual in treating thin rods. It is also easy to investigate the motion out of the plane of the disc. It is found that instability sets in just above the critical speed, but that by increasing the speed of rotation sufficiently the steady motion can be made stable in all cases. When the disc is perfectly centred the motion is always stable, but the frequency of vibration about the steady state passes through zero at the critical velocity.

On account of the criticisms† that can be levelled at the approximate theory of thin rods, it is desirable to seek some rigorous confirmation in the simple case of the unloaded shaft of circular cross-section. The present paper investigates some vibrations about the steady state of rotation round the axis on the lines of the work of Pochhammer‡ and Chree.§

* Cf. A. Föppl, *Civilingenieur* (1895), p. 334.

† See, for example, Love's *Elasticity*, Art. 258.

‡ L. Pochhammer, *Journal für Mathematik* (*Crelle*), Vol. 81 (1876), p. 324.

§ C. Chree, *Quarterly Journal of Mathematics*, Vol. 21 (1886), p. 287.

The General Equations.

Using the notation of Love's *Elasticity*, the equations of motion in cylindrical coordinates are

$$(\lambda+2\mu)\frac{\partial\Delta}{\partial r}-\frac{2\mu}{r}\frac{\partial\varpi_z}{\partial\theta}+2\mu\frac{\partial\varpi_\theta}{\partial z}=\rho f_r,$$

$$(\lambda+2\mu)\frac{1}{r}\frac{\partial\Delta}{\partial\theta}-2\mu\frac{\partial\varpi_r}{\partial z}+2\mu\frac{\partial\varpi_z}{\partial r}=\rho f_\theta,$$

$$(\lambda+2\mu)\frac{\partial\Delta}{\partial z}-\frac{2\mu}{r}\frac{\partial}{\partial r}(r\varpi_\theta)+\frac{2\mu}{r}\frac{\partial\varpi_r}{\partial\theta}=\rho f_z,$$

where f_r, f_θ, f_z are the accelerations in the directions of increase of r, θ, z, and the remaining quantities are given by

$$\Delta=\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial u_z}{\partial z},$$

$$2\varpi_r=\frac{1}{r}\frac{\partial u_z}{\partial\theta}-\frac{\partial u_\theta}{\partial z},$$

$$2\varpi_\theta=\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r},$$

$$2\varpi_z=\frac{1}{r}\frac{\partial(ru_\theta)}{\partial r}-\frac{1}{r}\frac{\partial u_r}{\partial\theta}.$$

Refer the motion to axes rotating with angular velocity ω, the axis of z remaining fixed. Then

$$f_r=\frac{\partial^2 u_r}{\partial t^2}-2\omega\frac{\partial u_\theta}{\partial t}-\omega^2(r+u_r),$$

$$f_\theta=\frac{\partial^2 u_\theta}{\partial t^2}+2\omega\frac{\partial u_r}{\partial t}-\omega^2 u_\theta,$$

$$f_z=\frac{\partial^2 u_z}{\partial t^2}.$$

The displacement includes that of the steady motion. It is easily seen that the equations of vibration about that state are obtained from the preceding by omitting the term $-\omega^2 r$ in f_r. Henceforth u_r, u_θ, u_z will be used for the *relative* displacement.

Try a solution

$$(u_r, u_\theta, u_z)=(R_1, R_2, R_3)\exp i(n\theta+\gamma z+pt),$$

where R_1, R_2, R_3 are functions of r only. They must satisfy the differential equations

$$(\lambda+2\mu)\left[\frac{d^2R_1}{dr^2}+\frac{1}{r}\frac{dR_1}{dr}-\frac{R_1}{r^2}+\frac{in}{r}\frac{dR_2}{dr}-\frac{in}{r}\frac{R_2}{r}+i\gamma\frac{dR_3}{dr}\right]$$
$$+\mu\left[-\gamma^2R_1-i\gamma\frac{dR_3}{dr}\right]-\mu in\left[\frac{1}{r}\frac{dR_2}{dr}+\frac{R_2}{r^2}-\frac{inR_1}{r^2}\right]$$
$$+\rho(p^2+\omega^2)R_1+2\rho\omega ipR_2=0,$$

$$(\lambda+2\mu)in\left[\frac{1}{r}\frac{dR_1}{dr}+\frac{R_1}{r^2}+\frac{inR_2}{r^2}+\frac{i\gamma R_3}{r}\right]+\mu\gamma\left[\frac{nR_3}{r}-\gamma R_2\right]-2\rho\omega ipR_1$$
$$+\mu\left[\frac{d^2R_2}{dr^2}+\frac{1}{r}\frac{dR_2}{dr}-\frac{R_2}{r^2}-\frac{in}{r}\frac{dR_1}{dr}+\frac{inR_1}{r^2}\right]+\rho(p^2+\omega^2)R_2=0,$$

$$(\lambda+2\mu)i\gamma\left[\frac{dR_1}{dr}+\frac{R_1}{r}+\frac{inR_2}{r}+i\gamma R_3\right]-\mu\left[i\gamma\frac{dR_1}{dr}+i\gamma\frac{R_1}{r}-\frac{d^2R_3}{dr^2}-\frac{1}{r}\frac{dR_3}{dr}\right]$$
$$-\mu n\left[\frac{nR_3}{r^2}-\gamma\frac{R_2}{r}\right]+\rho p^2R_3=0.$$

When $\omega=0$ and $n=0$ or 1 the forms of R_1, R_2, R_3 are contained in the papers already cited. An examination of the results suggested the following solution:—

$$R_1 = \quad A\,dJ_n(kr)/dr+DJ_n(kr)/r,$$
$$R_2 = -\,BdJ_n(kr)/dr+EJ_n(kr)/r,$$
$$R_3 = \quad CJ_n(kr).$$

In what follows we shall write

$$a\equiv(\rho p^2+\rho\omega^2-\mu\gamma^2)/(\lambda+2\mu),\qquad \beta\equiv(\rho p^2+\rho\omega^2-\mu\gamma^2)/\mu,$$
$$\delta\equiv\{\rho p^2-(\lambda+2\mu)\gamma^2\}/\mu,$$
$$b\equiv 2\rho\omega ip/(\lambda+2\mu),\qquad e\equiv 2\rho\omega ip/\mu,\qquad c\equiv(\lambda+\mu)i\gamma/(\lambda+2\mu),$$
$$f\equiv(\lambda+\mu)i\gamma/\mu.$$

The verification is carried out most easily by expressing everything in terms of dJ_n/dr and J_n, higher derivates being reduced by means of the formula

$$\frac{d^2J_n}{dr^2}=-\frac{1}{r}\frac{dJ_n}{dr}-k^2J_n+\frac{n^2J_n}{r^2}.$$

Putting the values in the original equations, we find

$$\left[2(D-inB)+\frac{(\lambda+\mu)in}{\lambda+2\mu}(E-inA)\right]\frac{1}{r^2}\frac{dJ_n}{dr}+(k^2A-aA+bB-cC)\frac{dJ_n}{dr}$$
$$+\left[2in(E-inA)-\frac{(\lambda+\mu)n^2}{\lambda+2\mu}(D-inB)\right]\frac{J_n}{r^3}$$
$$+\left[k^2D-aD-bE-\frac{(\lambda+\mu)in}{\lambda+2\mu}Bk^2\right]\frac{J_n}{r}=0,$$

$$\left[2(E-inA)-\frac{(\lambda+\mu)in}{\mu}(D-inB)\right]\frac{1}{r^2}\frac{dJ_n}{dr}+(eA-k^2B+\beta B)\frac{dJ_n}{dr}$$
$$+\left[-2in(D-inB)+\frac{(\lambda+\mu)n^2}{\mu}(E-inA)\right]\frac{J_n}{r^3}$$
$$+\left[eD+k^2E-\beta E-infC+\frac{(\lambda+\mu)in}{\mu}Ak^2\right]\frac{J_n}{r}=0,$$

$$f(D-inB)\frac{1}{r}\frac{dJ_n}{dr}+inf(E-inA)\frac{J_n}{r^2}-(fk^2A+k^2C-\delta C)J_n=0.$$

Equating the various coefficients to zero, we find a number of relations between the constants. These are all found to be consistent, and to be equivalent to the following :—

$$D=niB,\qquad E=niA,$$

and
$$\begin{aligned}(k^2-a)A+\quad & bB-\quad cC=0,\\ eA-(k^2-\beta)B\quad & =0,\\ fk^2A\quad & +(k^2-\delta)C=0.\end{aligned}$$

Thus k is given by the equation

$$(k^2-a)(k^2-\beta)(k^2-\delta)+be(k^2-\delta)+cfk^2(k^2-\beta)=0,$$

and
$$\frac{A}{c(k^2-\beta)}=\frac{B}{ce}=\frac{C}{k^4-k^2(a+\beta)+a\beta+be}=K \text{ (say)}.$$

General Method of Approximation.

The relation between p and γ is obtained from the condition that the stresses $\widehat{rr}$, $\widehat{r\theta}$, $\widehat{rz}$ should vanish at the cylindrical surface $r=a$. To each of the three effective values of k there corresponds an arbitrary quantity K, and the complete values of R_1, R_2, R_3 are obtained from the previous solutions by summation. In order to approximate to any desired

degree for a rod, we expand the Bessel functions in series to the required number of significant terms in r. Terms such as k^s, k^{s+2}, ... will appear, but by means of the determining cubic in k^2 we can get rid of all powers higher than the $(s+4)$-th. Thus the displacements can be expressed in the forms

$$R_1 = l_1 \Sigma K k^s + m_1 \Sigma K k^{s+2} + n_1 \Sigma K k^{s+4}, \ \ldots,$$

where the coefficients depend on r, n, γ, and p, and Σ indicates summation over the roots of the cubic. The conditions $\widehat{rr} = \widehat{r\theta} = \widehat{rz} = 0$ for $r = a$ give three equations from which the sums can be eliminated. The result is the required relation between a, n, γ and p; and since we can also find the ratios of the sums the character of the vibration is known.

The process is much more laborious than might appear at first sight, and even the second order approximation involves very long calculations.

The exponential factor will henceforth be omitted.

The Symmetrical Vibrations.

When $n = 0$ we have

$$R_1 = -\Sigma A k J_1(kr), \quad R_2 = \Sigma B k J_1(kr), \quad R_3 = C J_0(kr).$$

The components of stress can be found without difficulty. Taking the first non-vanishing term in the expansion of each quantity, we find, as a first approximation for a thin rod,

$$\begin{aligned}
R_1 &= -\tfrac{1}{2} r \Sigma A k^2 = -\tfrac{1}{2} rc\,(\Sigma K k^4 - \beta \Sigma K k^2), \\
R_2 &= \ \ \tfrac{1}{2} r \Sigma B k^2 = \ \ \tfrac{1}{2} rce\,(\Sigma K k^2), \\
R_3 &= \Sigma C = \Sigma K k^4 - (\alpha+\beta)\,\Sigma K k^2 + (\alpha\beta + be)\,\Sigma K, \\
\widehat{rr} &= -\Sigma[A k^2(\lambda+\mu) - i\gamma\lambda C] = (\lambda c + \mu c - i\gamma\lambda)\,\Sigma K k^4 \\
&\quad + (i\gamma\lambda\alpha + i\gamma\lambda\beta - \lambda c\beta - \mu c\beta)\,\Sigma K k^2 - i\gamma\lambda(\alpha\beta + be)\,\Sigma K, \\
\widehat{r\theta} &= -\tfrac{1}{8}\mu r^2 \Sigma B k^4 = -\tfrac{1}{8}\mu r^2 ce \Sigma K k^4, \\
\widehat{rz} &= -\tfrac{1}{2}\mu r \Sigma[i\gamma A k^2 + C k^2] = -\tfrac{1}{2}\mu r[(ic\gamma + \delta - cf)\,\Sigma K k^4 \\
&\quad + (\beta cf - ic\beta\gamma - a\delta - \beta\delta)\,\Sigma K k^2 + \delta(\alpha\beta + be)\,\Sigma K].
\end{aligned}$$

From the boundary conditions at the cylindrical surface we have therefore

$$\begin{vmatrix} \lambda c\beta + \mu c\beta - i\gamma\lambda a - i\gamma\lambda\beta, & i\gamma\lambda\,(\alpha\beta + be) \\ \beta cf - ic\beta\gamma - a\delta - \beta\delta, & \delta(\alpha\beta + be) \end{vmatrix} = 0.$$

The root $\alpha\beta + be = 0$ fails to determine the motion. The remaining roots are found to be

$$\beta = 0 \quad \text{and} \quad \lambda\delta + \mu\delta - i\gamma\lambda f - \lambda\gamma^2 = 0.$$

When $a\beta+be \neq 0$, we find

$$R_1 : R_2 : R_3 = r\beta\delta : re\delta : 2i\beta\,(\gamma+if).$$

Thus the root $\beta = 0$ gives purely torsional vibrations, for which

$$\rho(p^2+\omega^2) = \mu\gamma^2.$$

There is a large class of vibrations for which γ is real: for example, if l is the length of the rod the fixing of the ends makes γl an integral multiple of π. For these cases the rotating thin rod becomes unstable through the torsional displacement becoming excessive when $\rho\omega^2$ exceeds $\mu\gamma^2$.

The remaining root gives $\rho p^2 = E\gamma^2$, the same relation as for $\omega = 0$. The relative vibration is, however, not purely longitudinal.

Vibrations of Order Unity.

These correspond to flexural vibrations of the non-rotating bar, and the approximation must be taken to the second order. In order to get the final approximation correct we must start with three terms of the Bessel expansion. Omitting the factor 1/384, we write

$$J_1(kr) = 192kr-24k^3r^3+k^5r^5.$$

The quantity $\frac{1}{8}r^2$, or in the sequel $\frac{1}{8}a^2$, will be denoted by η. Hence we find

$$R_1 = 192\Sigma Ak+192i\Sigma Bk-72r^2\Sigma Ak^3-24ir^2\Sigma Bk^3+5r^4\Sigma Ak^5+ir^4\Sigma Bk^5,$$

$$R_2 = -192i\Sigma Ak-192\Sigma Bk-24ir^2\Sigma Ak^3+72r^2\Sigma Bk^3+ir^4\Sigma Ak^5-5r^4\Sigma Bk^5,$$

$$R_3 = 192r\Sigma Ck-24r^3\Sigma Ck^3+r^5\Sigma Ck^5,$$

and

$$-i\widehat{rr}/32r = (6\lambda+9\mu)\,i\,\Sigma Ak^3-3\mu\Sigma Bk^3+6\lambda\gamma\Sigma Ck$$
$$+\eta\{-(6\lambda+10\mu)\,i\,\Sigma Ak^5+2\mu\Sigma Bk^5-6\lambda\gamma\Sigma Ck^3\},$$

$$-\widehat{r\theta}/32\mu r = 3i\Sigma Ak^3-3\Sigma Bk^3+\eta\{-2i\Sigma Ak^5+4\Sigma Bk^5\},$$

$$\widehat{rz}/192\mu = i\gamma\Sigma Ak-\gamma\Sigma Bk+\Sigma Ck+\eta\{-3i\gamma\Sigma Ak^3+\gamma\Sigma Bk^3-3\Sigma Ck^3\}.$$

On substituting in terms of the K's there are four distinct sums. It is convenient to retain these and make use of a further equation which is at once derivable from the cubic in k^2, namely,

$$\Sigma Kk^7-(a+\beta+\delta-cf)\,\Sigma Kk^5+(a\beta+a\delta+\beta\delta+be-\beta cf)\Sigma Kk^3$$
$$-(a\beta+be)\delta\,\Sigma Kk = 0.$$

Then eliminating the four sums ΣKk^7, $-\Sigma Kk^5$, ΣKk^3, $-\Sigma Kk$ from this latter condition and the three boundary conditions, we have the period equation in the form

$$\begin{vmatrix} 1, & \alpha+\beta+\delta-cf, & \alpha\beta+\alpha\delta+\beta\delta+be-\beta cf, & (\alpha\beta+be)\,\delta \\ \eta(6\lambda\gamma+6\lambda ic+10\mu ic), & 6\lambda ic+9\mu ic+6\lambda\gamma+\eta(6\lambda ic\beta+10\mu ic\beta+6\lambda\gamma\alpha+6\lambda\gamma\beta+2\mu ce) & 6\lambda ic\beta+9\mu ic\beta+6\lambda\gamma\alpha+6\lambda\gamma\beta+3\mu ce+\eta(6\lambda\gamma\alpha\beta+6\lambda\gamma be), & 6\lambda\gamma(\alpha\beta+be) \\ 2ic\eta, & 3ic+\eta(2ic\beta^2+4ce), & 3ic\beta^2+3ce, & 0 \\ 3\eta, & 1+\eta(3\alpha+3\beta-3ic\gamma), & \alpha+\beta-ic\gamma+\eta(3\alpha\beta+3be-3ic\beta\gamma-ce\gamma), & \alpha\beta+be-ic\beta\gamma-ce\gamma \end{vmatrix} = 0$$

Considerable simplification is obtained by the ordinary methods of manipulation of determinants. In these processes the following identities are useful :

$$\frac{\alpha}{\beta}=\frac{\gamma+ic}{\gamma}=\frac{b}{e}=\frac{c}{f},$$

and about the simplest form into which the equation is directly reducible is

$$\begin{vmatrix} 1, & \alpha+\delta-cf, & \alpha\delta+be, & (\beta+ie)\,\delta \\ \eta(6\lambda\gamma+6\lambda ic+10\mu ic), & 6\lambda\gamma+6\lambda ic+9\mu ic+\eta(6\lambda\gamma\alpha+2\mu ce), & 6\lambda\gamma\alpha+3\mu ce+\eta(6\lambda\gamma be-2\mu ce\beta), & 6\lambda\gamma(\beta+ie) \\ 2ic\eta, & 3ic+4\eta ce, & 3ce-4\eta ce\beta, & 0 \\ 3\eta, & 1+\eta(3\alpha-3ic\gamma), & \alpha-ic\gamma+\eta(3be-ce\gamma), & \beta+ie-if\gamma \end{vmatrix} = 0.$$

At this stage it is desirable to insert the explicit values of α, β, b, ... Let us assume for the moment, what will be verified later, that the result of neglecting the terms in η is to give the result $p=\omega$. Then we may replace p by ω in the terms of the determinant which are or will be multiplied by η. There is a good deal of algebraic reduction to be performed, which leaves the equation in the form

$$\begin{vmatrix} \lambda+2\mu, & (\lambda+4\mu)\rho\omega^2-2\mu(\lambda+2\mu)\gamma^2, & \rho\omega^2+(\lambda+2\mu)\gamma^2, & -\rho\omega^2+(\lambda+2\mu)\gamma^2 \\ -2\mu\eta(2\lambda+5\mu), & -3\mu^2(\lambda+3\mu)+2\mu\eta(4\lambda\rho\omega^2-2\mu\rho\omega^2-3\lambda\mu\gamma^2), & -3\mu(\lambda-\mu)+2\eta(4\lambda\rho\omega^2-2\mu\rho\omega^2+\lambda\mu\gamma^2+\mu^2\gamma^2), & 6\lambda(\rho p^2+\rho\omega^2-2\rho\omega p-\mu\gamma^2) \\ 2\mu\eta, & 3\mu^2+8\mu\eta\rho\omega^2, & -3\mu+4\eta(2\rho\omega^2-\mu\gamma^2), & 0 \\ 3\eta(\lambda+2\mu), & \mu(\lambda+2\mu)+3\mu\eta(2\rho\omega^2+\lambda\gamma^2), & -\mu+\eta(6\rho\omega^2-\lambda\gamma^2-\mu\gamma^2), & \rho p^2+\rho\omega^2-2\rho\omega p+\lambda\gamma^2 \end{vmatrix} = 0.$$

The determinant is now expanded out and terms in η^2 rejected. The result is finally found to be

$$18\mu^2(\lambda+\mu)^2(\lambda+2\mu)\rho(p-\omega)^2 + 18\mu\eta\gamma^2[(2\lambda^3\mu+8\lambda^2\mu^2+10\lambda\mu^3+4\mu^4)\rho\omega^2 - \mu(\lambda+2\mu)\gamma^2(6\lambda^2\mu^2+10\lambda\mu^3+4\mu^4)] = 0,$$

which reduces to $$\rho\,(p-\omega)^2 = \tfrac{1}{4}a^2\gamma^2(E\gamma^2-\rho\omega^2).$$

The result $p = \omega$ for exceedingly thin rods is accordingly verified.

The ordinary critical speed is defined by the equation

$$\rho\omega^2 = \tfrac{1}{4}a^2E\gamma^4.$$

The equation for the frequencies of vibration at the critical speed, correct to terms in a^2, therefore becomes

$$\rho\,(p^2-2p\omega) = 0.$$

Thus one value of p becomes zero at the critical speed without being imaginary in the vicinity. Real instability sets in when $\rho\omega^2$ exceeds $E\gamma^2$.

Summary.

The cases treated correspond to well-defined modes of vibration of a non-rotating rod, and are here worked out to the same order as that which would be attained by the approximate theory of thin rods. For the shaft rotating centrically the ordinary critical speed is found to have no influence in converting stability into instability. On the other hand, the analysis indicates two other critical speeds, which are numerically much greater. These occur when the time of steady rotation coincides with the natural periods of torsional and longitudinal vibrations about the position of rest, and the latter speed affects the stability of the quasi-flexural but not that of the quasi-longitudinal vibration.

A NOTE ON THE PROPERTY OF BEING A DIFFERENTIAL COEFFICIENT

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1. Recent research has provided us* with a set of necessary and sufficient conditions that a function may be an indefinite integral, in the generalised sense, of another function, and the way has thus been opened to important developments. The corresponding, much more difficult, problem of determining necessary and sufficient conditions that a function may be a differential coefficient, has barely been mooted; indeed, though we know a number of necessary conditions, no set even of sufficient conditions has to my knowledge ever been formulated, except that involved in the obvious statement that a continuous function is a differential coefficient. The necessary conditions in question are of considerable importance and interest. A function which is a differential coefficient has, in fact, various striking properties. It must be pointwise discontinuous with respect to

* H. Lebesgue, "Leçons sur l'intégration," p. 129, footnote (statement only); "Sur la recherche des fonctions primitives par l'intégration," 1907, *Rend. di Torino dei Lincei*, Vol. XXI, pp. 206–287; "Sur les Intégrales singulières," 1909, *Ann. de Toulouse*, 3rd Series, Vol. I, p. 43. G. Vitali, "Sulle funzioni integrali," 1905, *Atti di Torino*, Vol. XL, pp. 1021–1034.

For properties of functions which are integrals, reference may be made to three recent papers, viz., "Functions of Bounded Variation" (1910), *Quarterly Journal;* and "On Semi-Integrals and Oscillating Successions of Functions" (1910), and "On the Existence of a Differential Coefficient" (1910), the two latter presented to the London Mathematical Society.

Among the properties not there alluded to, we may specially mention the following:—The product of two integrals is an integral, and so is also their quotient, provided it is bounded. If $f(x)$ is an integral, and $g(x)$ is also an integral, then $f[g(x)]$ is an integral. If u and v are integrals, we have, in all cases,

$$uv = \int u \frac{dv}{dx}\, dx + \int v \frac{du}{dx} dx,$$

the integrations being over the sets of points at which the respective differential coefficients exist and are finite.

every perfect set; it can have no discontinuity of the first kind; it assumes in every interval all values between its upper and lower bounds in that interval; its value at any point is one of the limits on both sides of the values in the neighbourhood; its upper and lower bounds, when finite, are unaltered, if we omit any set of points of content zero, and, in the general case, are unaltered, if we omit the values at any countable set of points; the points at which it is infinite form an inner limiting set of content zero. From these necessary conditions we are able to deduce much valuable information as to when a function is certainly not a differential coefficient. They enable us to realise the very special characteristics of a function which is a differential coefficient. It is clear that, for example, a function which is a differential coefficient ceases to be a differential coefficient if its value be altered at a single point. These conditions do not, however, render us any material assistance, even in answering the simple question as to whether the product of two differential coefficients is a differential coefficient, and this not even in the special case in which one of the differential coefficients is a continuous function.

It is with this last-named case that the present paper is concerned. In view of the importance of the problem in the theory of the differentiation of infinite series and of improper integrals—where the property of being a differential coefficient presents itself naturally as the necessary and sufficient condition which must hold in order that such differentiation may, on certain assumptions, be permissible—as well as in other applications, it is hoped that the results here obtained will be regarded as of interest. These are in the main as follows:—

A function which is a differential coefficient remains a differential coefficient when multiplied by a continuous function, provided only that this continuous function possesses a bounded differential coefficient; or, more generally, that this continuous function is an integral, and possesses a differential coefficient, which is bounded either above or below.

Again, a function which, being a differential coefficient, is also finite and summable, remains a finite and summable differential coefficient when multiplied by a continuous function, provided only that this continuous function has bounded variation.

We have also the associated theorem that a function which is the differential coefficient of a function of bounded variation remains the differential coefficient of a function of bounded variation when multiplied by a continuous function, provided only that this continuous function possesses a finite and summable differential coefficient.

Finally, we have the theorems associated to the first mentioned, namely, that a function, which is a bounded differential coefficient, or

has as primitive function an integral, and is bounded either above or below, remains a differential coefficient of the same kind when multiplied by any continuous function.

It will be seen that only the fringe of the main question of finding sufficient conditions that a function may be a differential coefficient has been touched. Partial, however, as the results obtained may from this point of view appear, they would seem from the nature of the reasoning employed to have a certain finality. Examples illustrating the limitations imposed by our theorems are desirable, but the work of constructing such examples is both difficult and laborious.

2. We proceed to prove the theorems in a somewhat different order from that in which it was convenient to state them in the introduction.

In the first theorem we employ the method of change of variable, and purposely, for the sake of simplicity, give the statement of the theorem in a somewhat less general form than is possible.

THEOREM 1. — *If $f'(x)$ is a bounded differential coefficient, then $f'(x)\,g(x)$ is also a bounded differential coefficient, provided only that $g(x)$ is a continuous function.*

Let A be any quantity *less* than the lower bound of $f'(x)$ in the interval considered. Then, writing

$$h(x) = f(x) + Ax,$$

we see that $h'(x)$ is everywhere *greater* than zero, so that $h(x)$ is monotone increasing and nowhere stationary.

Put then $h(x) = t$; then $x = h^{-1}(t)$ is a monotone increasing continuous function of t. Hence $g\{h^{-1}(t)\}$ is a continuous function of t.

Writing therefore

$$F(x) = \int^x f'(x)\,g(x)\,dx,$$

we have

$$F(x) = \int^x h'(x)\,g(x)\,dx - A\int^x g(x)\,dx. \qquad (1)$$

Bearing in mind that $h'(x)$ is bounded, so that it is the differential coefficient of its integral, and this integral is $h(x)$, we may in the first term

on the right-hand side of (1) change the variable from x to t, and thus get

$$\int^{t} g\{h^{-1}(t)\}\,dt,$$

of which the differential coefficient with respect to t is, of course, $g\{h^{-1}(t)\}$, since this function is continuous, *i.e.*, it is $g(x)$. It has accordingly a differential coefficient with respect to x, namely $g(x)\,dt/dx$, *i.e.*, $g(x)h'(x)$.

Also $A\int^{x} g(x)\,dx$ has a differential coefficient, namely $Ag(x)$, since this function is continuous. Hence, finally, we see from (1) that $F(x)$ has a differential coefficient, viz.,

$$F'(x) = g(x)\,h'(x) - Ag(x) = g(x)f'(x),$$

and this proves the required theorem.

3. In proving the following theorem, we require a different method, involving the use of the Second Theorem of the Mean in Du Bois Reymond's form.

THEOREM 2.—*If $f'(x)$ is a finite and summable, but not necessarily bounded, differential coefficient, then $f'(x)\,g(x)$ is a finite and summable differential coefficient, provided only that $g(x)$ is a continuous function of bounded variation.*

If we can prove this theorem when $g(x)$ is a monotone decreasing function, we can, since every function of bounded variation is the difference of two such functions, clearly deduce the required result by subtraction in the general case. Suppose then $g(x)$ to be monotone decreasing, then, by the Second Theorem of the Mean in its most general form, we may write

$$\int_{x}^{x+h} f'(t)\,g(t)\,dx = g(x)[f(x+X)-f(x)]+g(x+h)[f(x+h)-f(x+X)],$$

where $$0 \leqslant X \leqslant h.$$

In fact, since $f'(x)$ is finite and summable, its integral is $f(x)$. Hence, if we denote $\int^{x} f'(t)\,g(t)\,dt$ by $F(x)$, we have, rearranging the right-hand

side, and dividing both sides by h,

$$\frac{F(x+h)-F(x)}{h} = g(x+h)\frac{f(x+h)-f(x)}{h} + z[g(x)-g(x+h)]\frac{f(x+X)-f(x)}{X},$$

where $$0 \leqslant z \leqslant 1.$$

The first term on the right-hand side has clearly the unique limit $g(x)f'(x)$ as h diminishes to zero. The second term has, on the other hand, the unique limit zero, since $g(x)$ is continuous, and $f'(x)$ is *finite*. Hence $F(x)$ has a differential coefficient, and this differential coefficient is $g(x)f'(x)$, which proves the theorem.

4. Before proceeding to the next theorems, it will be convenient to prove the rule for the differentiation of a product in such a form as to include the possibility of one of the differential coefficients being infinite with a definite sign. We find that the theorem still holds in this case even when the remaining function is zero at the point at which the differential coefficient in question is infinite, provided only that we ascribe the value zero, in this case, to their product $0 \times \infty$. We have, in fact,

$$f(x+h)\frac{g(x+h)-g(x)}{h} + g(x)\frac{f(x+h)-f(x)}{h} = \frac{f(x+h)\,g(x+h)-f(x)\,g(x)}{h},$$

of which the second term on the left-hand side is permanently zero at a point at which $g(x)$ is zero, and therefore has the unique limit zero, while at a point at which $g(x)$ is not zero, the term in question has a unique limit, whether $f'(x)$ be infinite with determinate sign, or finite. It is plain, therefore, that the theorem holds, with the meaning attached to the indeterminate form. It is in this sense that the following theorems hold, should such an indeterminate form arise.

5. We can now at once prove the following theorem :—

THEOREM 3.—*If $f'(x)$ is the differential coefficient of a function of bounded variation, then [whether $f'(x)$ is finite or not] $f'(x)g(x)$ is the differential coefficient of a function of bounded variation, provided only that $g(x)$ possesses a finite and summable differential coefficient [so that $g(x)$ is, of course, continuous, of bounded variation, and an integral].*

In fact, since $f'(x)$ and $g'(x)$ both exist, and $g'(x)$ is finite,

$$f'(x)\,g(x) = \frac{d}{dx}\{f(x)\,g(x)\} - f(x)\,g'(x). \qquad (1)$$

Now $f(x)\,g'(x)$ is, by hypothesis, the product of a function of bounded variation and a finite and summable differential coefficient, and is therefore, by Theorem 2, a finite and summable differential coefficient, and accordingly also the differential coefficient of its integral, *i.e.*, of a function of bounded variation.

Also $g(x)$ is the integral of its differential coefficient, and therefore has bounded variation. Therefore, since both $f(x)$ and $g(x)$ have bounded variation, so has $f(x)\,g(x)$.

Thus $f'(x)\,g(x)$ is the difference of the differential coefficients of two functions of bounded variation, and is therefore itself the differential coefficient of a function of bounded variation.

6. If we apply the method of the preceding article to transform the theorem of § 2, we obtain the following theorem :—

THEOREM 4.—*If $f'(x)$ is a differential coefficient* [*whether finite or not, summable or not, and whether $f(x)$ be of bounded variation or not*], *then $f'(x)\,g(x)$ is a differential coefficient, provided only that $g(x)$ possesses a bounded differential coefficient* [*so that $g(x)$ is, of course, continuous, of bounded variation, and an integral*].

For $f(x)$ being, of course, continuous, it follows, by Theorem 1, that $f(x)\,g'(x)$ is a differential coefficient. Also, since $f'(x)$ and $g'(x)$ both exist, and $g(x)$ is finite,

$$f'(x)\,g(x) = \frac{d}{dx}[f(x)\,g(x)] - f(x)\,g'(x).$$

Thus $f'(x)\,g(x)$ is the excess of the differential coefficient of $f(x)\,g(x)$ over another differential coefficient, that is, it is itself a differential coefficient.

7. We now return to equation (1) of § 2, and remark that, like the reasoning that precedes it, this continues to hold if $f'(x)$ is bounded in one sense only. The change of variable carried out in the first integral of this equation is, however, not always allowable; for it to be so, $h(x)$ must be assumed to be the integral of $h'(x)$, which is no longer necessarily the case. If, however, $f(x)$, and therefore $h(x)$, is an integral, $h(x)$ is necessarily the integral of its differential coefficient. Thus we have at once the theorem :—

THEOREM 5.—*If $f'(x)$ is bounded either above or below, and $f(x)$ is an*

integral, then $f'(x)\,g(x)$ is a differential coefficient, provided only that $g(x)$ is a continuous function.

The proof is, in fact, clearly complete, except in the possible case of a zero of $g(x)$ coinciding with an infinity of $f'(x)$. Should this happen, the theorem would still be true if we substituted $g(x)+A$ for $g(x)$, A being suitably chosen. Thus we should have

$$f'(x)[g(x)+A] = \frac{d}{dx}\int^x f'(x)\,[g(x)+A]\,dx.$$

Hence, at a point at which $g(x)$ is zero, this gives

$$Af'(x) = \frac{d}{dx}\left[\int^x f(x)\,g(x)\,dx + Af(x)\right];$$

and therefore at such a point $\dfrac{d}{dx}\displaystyle\int^x f(x)\,g(x)\,dx$ is zero, and is therefore equal to $f'(x)\,g(x)$, provided we ascribe the value zero to this expression although it now assumes the indeterminate form $\infty \times 0$. With this understanding then, the theorem has been shewn to hold always.

As a corollary to this theorem we have the following, which, of course, includes Theorem 1 as a particular case :—

COR.—*If $f'(x)$ is finite and bounded either above or below, then $f'(x)\,g(x)$ is a differential coefficient, provided only that $g(x)$ is a continuous function.*

8. Just as in §§ 5 and 6, we transform the theorem of the preceding article into an associated theorem, in our case the following :—

THEOREM 6.—*Whatever be the nature of the differential coefficient $f'(x)$, it remains a differential coefficient when multiplied by any continuous function which is an integral and possesses a differential coefficient which is bounded either above or below, and in particular by any continuous function which possesses a differential coefficient which is finite and bounded either above or below.*

9. We now pass to the case when the differential coefficient, though finite and summable, is unbounded both above and below, and suppose that we wish to multiply it by a perfectly general continuous function.

We may in this case write

$$f(x) = P(x) - N(x),$$

where $P(x)$ is the positive variation of $f(x)$, and, besides being a monotone increasing continuous function, is an integral, viz., the integral of $f'(x)$ over the set of points at which $f'(x)$ is positive, and where $N(x)$ is the absolute value of the negative variation, and possesses similar properties to $P(x)$. Evidently $P'(x)$ and $N'(x)$, being always positive, are bounded below; we are thus able to apply Theorem 5 to each separately, and obtain equations

$$P'(x)\, g(x) = \frac{d}{dx} \int^x P(x)\, g(x)\, dx,$$

$$N'(x)\, g(x) = \frac{d}{dx} \int^x N(x)\, g(x)\, dx.$$

Each of these equations involves, it should be remarked, two statements, the first of which asserts that the differential coefficient on the right-hand side exists, and the second that the value of this differential coofficient is the left-hand side.

Subtracting these two equations we can then assert that, at a point where $P'(x)$ and $N'(x)$ are finite,

$$\int^x f(x)\, g(x)\, dx$$

has a finite differential coefficient, and that this differential coefficient is

$$f'(x)\, g(x).$$

But we are not able to assert the existence of a finite differential coefficient at a point where $P'(x)$, $N'(x)$ are both infinite, and still less that, if it should happen to exist at such a point, its value is $f'(x)\, g(x)$.

Now though we are sure that, since $P(x)$ and $N(x)$ are integrals, $P'(x)$ and $N'(x)$ exist everywhere, there is no theorem that assures us that these differential coefficients are everywhere finite. All that we can conclude is that they are infinite together with the same sign, and that the points at which this happens form at most a set of content zero. Hence we have the following theorem:—

THEOREM 7.—*If $f'(x)$ is a finite summable function, and $g(x)$ any continuous function, then $\int^x f'(x)\, g(x)\, dx$, which necessarily exists, and there-*

fore possesses a finite differential coefficient except at a set of content zero, has $f'(x)\,g(x)$ for its differential coefficient, except at a set of content zero (which there is nothing to shew coincides with the former set of content zero).

Note.—The preceding argument requires only slight modification if we suppose that $f'(x)$ is summable, and finite except at a reducible set of points. The only difference is, in fact, that we can no longer assert that $P'(x)$ and $N'(x)$ should be infinite together, though this may be the case.

ON NON-INTEGRAL ORDERS OF SUMMABILITY OF SERIES AND INTEGRALS

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Introduction.

1. In this paper I propose to develop the theory of Cesàro's mean value process for summing a series, when non-integral "orders of summability" are considered.

Disregarding the way in which Cesàro originally built up his method, I take as the basis of the extended theory the general formula which is arrived at* for the r-th mean, viz., $S_n^{(r)}/A_n^{(r)}$, where

$$S_n^{(r)} = s_n + \binom{r}{1} s_{n-1} + \binom{r+1}{2} s_{n-2} + \ldots + \binom{r+n-1}{n} s_0 \qquad (1)$$

$$= u_n + \binom{r+1}{1} u_{n-1} + \binom{r+2}{2} u_{n-2} + \ldots + \binom{r+n}{n} u_0, \qquad (2)$$

$$A_n^{(r)} = \binom{r+n}{n} = \frac{(r+1)(r+2)\ \ldots\ (r+n)}{n!}. \qquad (3)$$

In these formulæ r may be supposed to have any real or complex values whatever, save negative integral values, which for obvious reasons are throughout excluded. Then, if

$$S_n^{(r)}/A_n^{(r)}$$

tends to a definite limit L_r as $n \to \infty$, the series will be said to be

* Bromwich, *Infinite Series*, §§ 122–128.

summable by Cesàro's method of order r, or, briefly, *summable* (*Cr*).* If no limit exists, but the same function of n oscillates finitely as $n \to \infty$, Σu_n is said to be *finite* (*Cr*).*

When r is restricted to be zero or a positive integer, it is well known that if Σu_n is summable (*Cr*), it is also summable ($C\overline{r+1}$), with the same "sum"; also that $u_n/n^r \to 0$ as $n \to \infty$, and $\Sigma u_n x^n \to L_r$ as $x \to 1$.

Moreover, if Σu_n is summable (*Cr*) and Σv_n is summable (*Cs*), r and s being zero, or positive integers, then the product series Σw_n, formed according to Cauchy's rule (*i.e.*, $w_n = u_0 v_n + u_1 v_{n-1} + \ldots + u_{n-1} v_1 + u_n v_0$) is summable ($C\overline{r+s+1}$), with a "sum" equal to the product of the sums of Σu_n and Σv_n.

In the present paper I have confined myself to real values of r,† positive or negative, integral or not. It will appear that the properties of the extended method of summation are quite different on the two sides of the point $r = -1$. All the ordinary properties which hold for positive integral values of r also hold for non-integral values of r which exceed -1. The marked changes in the formal theory for lower values of r are illustrated by the fact that *divergent* series exist (see § 7) which are summable (*Cr*), r being < -1, with the sum *zero*.

2. The extension of Cesàro's method to positive non-integral orders of r is of little advantage in assisting us to find the actual "sum" of a non-convergent series; it is almost invariably simpler to do this by using the Cesàro mean value process of some *integral* order (in §§ 4, 5 it is shewn that the sum is the same in either case). The value of non-integral orders of summability lies in the closer information which we may gain by their use concerning the *degree* or *amount* (speaking in a general way) of the non-convergence of series; and by which we can more narrowly predict the behaviour of a series when multiplied by other series, or by a sequence of convergence factors (see §§ 10, 13–17, *et seq.*).

When the order of summability is negative, on the other hand, Cesàro's method (for $r > -1$) is wholly inapplicable to non-convergent series. It is useful only for convergent series, and here again, not because it enables

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* This convenient notation was introduced by Mr. Hardy in a paper which appeared in the *Proc. London Math. Soc.*, Ser. 2, Vol. 6, p. 257. The notion of a series being *finite* (*Cr*) is due to M. Bohr, *Comptes Rendus*, 1909.

† Save in dealing with the class of series $\Sigma n^s e^{in}$, in § 23 and § 26.

us to find their sum* (which may be supposed already known), but because of the insight into the nature of the convergence of series, which the method enables us to obtain.

Note.—The idea of non-integral orders of summability (with Cesàro's method) is not entirely novel, as I discovered after this paper was written. Dr. Marcel Riesz had stated several interesting theorems in the *Comptes Rendus*, June, 1909, in which such "orders" are considered. As he has not yet published more detailed statements and proofs of his theorems, it is difficult to say how far our work has coincided; but, so far as I am aware, this does not extend beyond the theorems of §§ 17, 18. My attention was drawn by Mr. Hardy to Dr. Riesz's writings after my own work was completed.

In two recent letters to me (December, 1910, and March, 1911) Dr. Riesz mentions that Knopp has also considered such orders of summability as are here dealt with (*Inaugural Dissertation*, Berlin, 1907, and *Sitzungsberichte der Berliner Math. Gesellschaft*, 1907; *Archiv der Math. u. Phys.*, Bd. 12). I have, unfortunately, not had an opportunity of reading Dr. Knopp's memoirs.

The extension of Cesàro's method to convergent series, by using negative values of r, is a particular case of the application of methods of summation,† in general, to convergent series, a treatment of the latter which appears never to have been considered before. This idea, amongst other recent extensions, in the theory of summability (in particular, Dr. Riesz's elegant and powerful method of summation‡) is discussed more generally in a paper by Mr. Hardy and myself (written since the present paper was completed) on "A General View of the Theory of Summable Series."§

Part I of this paper deals with the general theory of the extended mean value process, for all real values of r. Parts II and III contain those parts of the theory (and some of its applications) which relate more particularly to non-convergent and convergent series respectively.

Part IV contains a few theorems in the closely parallel theory of summable integrals; they are naturally suggested by the corresponding theorems for series, but their proofs take a more elegant form, and they were consequently thought worthy of inclusion.

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* Of course the "sum" found by the extended Cesàro method must evidently agree with the sum as ordinarily defined, if serious complications are to be avoided in the use of the method.

† We refer to methods which are applicable *only* to convergent series; of course most methods which will sum non-convergent series will also sum convergent series, but no new information concerning the latter is gained from that fact.

‡ See § 3 of this paper for a brief description of this method, and reference (9), § 2, for the memoirs in which the method was first described.

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For convenience I here append a list* of the memoirs to which I shall have occasion to refer. This list does not purport to be at all a complete bibliography of the subject.† Further, I would make a general reference to Dr. Bromwich's *Theory of Infinite Series*, §§ 122–128, where an account of Cesàro's method for positive integral orders of summability is given; hereafter I shall refer to this treatise by the initials (*I. S.*, p. ...).

PART I.

§§ 3. Preliminary formulæ.
4, 5. The condition of consistency.
6. On the limit of $\Sigma u_n x^n$ as $x \to 1$.
7. Divergent series *may* be summable.
8. General condition of consistency.
9. The index of summability.
10. The generalized multiplication theorem.
11. If Σu_n is summable (Cr), $r > -1$, $\lim_{n\to\infty} \frac{u_n}{w} = 0$.

3. The following formulæ (4)–(6) will constantly be needed. They hold good whether r is integral or not,

$$\Sigma S_n^{(r)} x^n = (1-x)^{-r} \Sigma s_n x^n \qquad (4)$$

$$= (1-x)^{-(r+1)} \Sigma u_n x^n, \qquad (5)$$

$$\Sigma A_n^{(r)} x^n = (1-x)^{-(r+1)}. \qquad (6)$$

All these series converge absolutely for $|x| < 1$, if Σu_n is summable (Cr) or finite (Cr).

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* 1. E. Cesàro.—*Bulletin des Sciences Mathématiques*, 2e série, t. 14.
2. L. Fejér.—*Math. Annalen*, Bd. 58, 1903.
3. G. H. Hardy.—*Proc. London Math. Soc.*, Ser. 2, Vol. 4, p. 247.
4. C. N. Moore.—*Trans. Amer. Math. Soc.*, Vol. 8, 1907.
5. G. H. Hardy.—*Math. Annalen*, Bd. 64, 1907.
6. T. J. I'A. Bromwich.—*Math. Annalen*, Bd. 65, 1907.
7. G. H. Hardy.—*Proc. London Math. Soc.*, Ser. 2, Vol. 6, p. 255, 1908.
8. H. Bohr.—*Comptes Rendus*, January 11th, 1909.
9. M. Riesz.—*Comptes Rendus*, June, July, November, 1909.
10. G. H. Hardy.—*Proc. London Math. Soc.*, November, 1909.
11. L. Fejér.—*Comptes Rendus*, December, 1900; April, 1902.
12. L. Fejér.—*Mathematikai és Physikai Lapok*, 1902.
13. L. Fejér.—*Math. Annalen*, Bd. 58, 1904.
14. G. H. Hardy and S. Chapman.—*Quarterly Journal*, 1911.

† A more extended list of cognate memoirs will be found in the paper (14) of the preceding footnote.

From (5) it follows that

$$\Sigma u_n x^n = (1-x)^{r+1} \Sigma S_n^{(r)} x^n,$$

and hence
$$u_n = S_n^{(r)} - \binom{n+1}{1} S_{n-1}^{(r)} + \binom{r+1}{2} S_{n-2}^{(r)} - \ldots, \tag{7}$$

where (for $n \geqslant r+2$) there are $r+2$ or $n+1$ terms according as r is or is not a positive integer.

It is important to notice that, if $r > -1$, $A_n^{(r)}$ is positive; if

$$-\overline{p+1} < r < -p,$$

p being a positive integer, then $A_n^{(r)}$ alternates in sign as n takes the values $0, 1, \ldots, p+1$, after which its sign is constant, that of $(-1)^p$. Also, if $r > 0$, $A_n^{(r)}$ steadily increases as $n \to \infty$; if $-1 < r < 0$, $A_n^{(r)}$ steadily decreases to 0 as $n \to \infty$.

Evidently
$$A_n^{(r)} = p_{n,r}\, n^r / \Gamma(1+r),$$

where, as $n \to \infty$, r remaining constant,

$$p_{n,r} \to 1.$$

Hence, since
$$\binom{r+m}{m} = A_m^{(r)},$$

$$\frac{S_n^{(r)}}{A_n^{(r)}} = \sum_{m=0}^{n-1} u_m \frac{A_{n-m}^{(r)}}{A_n^{(r)}} + \frac{u_n}{A_n^{(r)}}.$$

Therefore
$$\lim_{n\to\infty} \frac{S_n^{(r)}}{A_n^{(r)}} = \lim_{n\to\infty} \frac{1}{p_{n,r}} \sum_{m=0}^{n-1} p_{n-m,r}\, u_m \left(1-\frac{m}{n}\right)^r + \lim_{n\to\infty} \frac{u_n}{A_n^{(r)}}$$

$$= \lim_{n\to\infty} \sum_{m=0}^{n-1} p_{n-m,r}\, u_m \left(1-\frac{m}{n}\right)^r,$$

since
$$\lim p_{n,r} = 1, \quad \lim u_n/n^r = 0 \quad \text{(see § 11)}.$$

Now let us consider the method of summation invented by Dr. Riesz. Here the "sum" of a series is defined by the limit, as n tends *continuously* to infinity, of

$$\Sigma_n^{(r)} = \sum_{m=0}^{[n]} u_m \left(1-\frac{\lambda(m)}{\lambda(n)}\right)^r;$$

where $[n]$ denotes the greatest integer less than n, $\lambda(n)$ is a monotonic positive function of n (such as n or $\log n$) tending to infinity with n; and r is the order of summability (R, λ)—in Mr. Hardy's notation. If the limit exists, the series is said to be *summable by Riesz's method of order r*, or, briefly, *summable* (R, λ, r).

When $\lambda(n) = n$,* $\Sigma_m^{(r)}$ takes the form

$$\sum_{m=0}^{[n]} u_m \left(1 - \frac{m}{n}\right)^r,$$

the limit of which (since $p_{n,r} \to 1$ as $n \to \infty$) may be expected to be almost identical with that of $S_n^{(r)}/A_n^{(r)}$. As a matter of fact, the existence of either limit follows from and involves that of the other, and the two limiting values are identical; though this would not be the case if n in $\Sigma_n^{(r)}$ were only required to tend to infinity through positive integral values (as in $S_n^{(r)}/A_n^{(r)}$), instead of continuously.†

Thus Cesàro's method, and that particular case of Riesz's method for which $\lambda(n) = n$, are completely identical in range. *Therefore all the theorems proved in this paper for series summable* (Cr) *hold good for series summable* (R, n, r). Many general theorems seem easier to prove directly for Cesàro's than for Riesz's method; but it often happens that (as in § 24) the latter method is the easier to apply in order to find the sum and summability of particular series.

4. From (4)–(6) it appears that

$$\left.\begin{aligned} S_n^{(r)} &= S_n^{(r+1)} - S_{n-1}^{(r+1)} \\ A_n^{(r)} &= A_n^{(r+1)} - A_{n-1}^{(r+1)} \end{aligned}\right\}, \tag{8}$$

for all values of r. By Stolz's extension‡ of Cauchy's theorem it at once follows that, *if* Σu_n *is summable* (Cr), *it is also summable* $(C\,\overline{r+1})$, *provided that* $r > -1$.

Stolz's theorem is that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n},$$

provided that the latter limit exists, while b_n steadily increases to infinity, as $n \to \infty$. Put

$$a_n = S_n^{(r+1)}, \quad b_n = A_n^{(r+1)};$$

then b_n steadily increases with n if $r > -1$.

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* We shall here confine ourselves to this case. For a treatment of the general case, which has very interesting properties, the reader is referred to (9) and (14) of our list.

† For this information I am indebted to the courtesy of Dr. Riesz, to whom the stated theorem is due. In the original announcement of the theorem in the *Comptes Rendus*, July and November, 1909, there was some ambiguity, owing to the crowding-out of certain important explanatory remarks. Dr. Riesz has apparently considered only real positive values of r; but his proof can be extended down to $r > -1$. In a shortly forthcoming article in the *Comptes Rendus*, Dr. Riesz intends to prove the theorem in full.

‡ Bromwich, *Infinite Series*, App., § 152, II.

5. Further, *if* Σu_n *is summable* (Cr), *then it is also summable* $(C\overline{r+1})$, *with the same sum, for* $r<-1$, *provided that* $\Sigma u_n x^n$ *has a finite upper limit as* $x\to 1$.

In particular, the theorem is true if, as in the case of all convergent series, $\Sigma u_n x^n$ tends to a finite limiting value as $x\to 1$.

By a limit theorem similar in form to that of Stolz (*I. S.*, § 152, I),

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n},$$

provided that the latter limit exists, that b_n steadily decreases to zero, and that $a_n\to 0$. The existence of the second limit above involves the existence of $\lim_{n\to\infty} a_n$, for

$$|a_{n+1}-a_n|<K(b_n-b_{n+1});$$

and therefore $$|a_{n+m}-a_n|<K(b_n-b_{n+m})<Kb_n,$$

which (since $b_n\to 0$ as $n\to\infty$) is a sufficient condition for the existence of $\lim a_n$.

In this limit theorem put $a_n=S_n^{(r+1)}$, $b_n=A_n^{(r+1)}$; b_n satisfies the stated condition provided $r<-1$, and the right-hand limit exists. Hence $\lim S_n^{(r+1)}$ exists, *i.e.*, the series $\sum_{n=0}^{\infty} S_n^{(r)}$ is convergent. Thus, by Abel's theorem,

$$\lim_{n\to\infty} S_n^{(r+1)}=\lim_{x\to 1}\sum_{n=0}^{\infty} S_n^{(r)}x^n$$

$$=\lim_{x\to 1}(1-x)^{-(r+1)}\Sigma u_n x^n,$$

by (5). If $\overline{\lim_{x\to 1}}\ \Sigma u_n x^n$ is finite, and $r<-1$, it follows that

$$\lim_{n\to\infty} S_n^{(r+1)}=0,$$

and all the conditions of the limit theorem are fulfilled. Consequently $\lim_{n\to\infty} S_n^{(r+1)}/A_n^{(r+1)}$ is equal to $\lim_{n\to\infty} S_n^{(r)}/A_n^{(r)}$, which proves the theorem.

6. The theorem that, *if* Σu_n *is summable* (Cr), *then*

$$\lim_{x\to 1}\Sigma u_n x^n=L_r,$$

was proved by Dr. Bromwich.* His proof, of course, dealt only with positive integral values of r, but also applies to general values, under

* *Math. Annalen*, Bd. 65, 1907.

certain limitations. It rested on a well known theorem due to Cesàro, viz.,

$$\lim_{x\to 1} \frac{\sum_0^\infty a_n x^n}{\sum_0^\infty b_n x^n} = \lim_{n\to\infty} \frac{\sum_0^n a_n}{\sum_0^n b_n},$$

provided that the latter limit exists, Σb_n being a series of positive terms.

By (5) and (6), it follows that

$$\sum_0^\infty u_n x^n = \frac{\sum_0^\infty S_n^{(r)} x^n}{\sum_0^\infty A_n^{(r)} x^n}.$$

Hence
$$\lim_{x\to 1} \Sigma u_n x^n = \lim_{n\to\infty} \frac{\sum_0^n S_n^{(r)}}{\sum_0^n A_n^{(r)}} = \lim_{n\to\infty} \frac{S_n^{(r+1)}}{A_n^{(r+1)}} = L_{r+1},$$

provided that the latter limit exists, and that $A_n^{(r+1)}$ is positive for all values of n.

Now, if L_r exists, and $r > -1$, both these conditions are satisfied, and $L_r = L_{r+1}$. Therefore, *if* Σu_n *is summable* (Cr) *and* $r > -1$,

$$\lim_{x\to 1} \Sigma u_n x^n = L_r.$$

The condition $r > -1$ is sufficient but not necessary, because the theorem holds true for *convergent* series which are summable (Cr), whether r is $>$ or < -1; by § 5, L_{r+1} for such a series is equal to L_r.

7. Since, if L_r exists for any value of $r > -1$, $\Sigma u_n x_n \to L_r$ as $x \to 1$, it follows that only convergent and oscillatory (not divergent) series are summable (Cr) for $r > -1$. But it is easy to give an example of a divergent series for which $L_r = 0$ for suitable values of $r < -1$, and for which, of course, $\Sigma u_n x^n \to \infty$ as $x \to 1$.

The simplest instance is the series for which $u_n = A_n^{(k)}$, where k is positive but not integral. Then

$$\Sigma S_n^{(-k+2)} x^n = (1-x)^{(-k-2+1)} \Sigma u_n x^n = (1-x)^{k+1} (1-x)^{-(k+1)} = 1.$$

Thus $S_n^{(-k+2)} = 0$ for $n \geqslant 1$. Now $A_n^{(-k+2)} \neq 0$ for any value of n, though $\to 0$ as $n \to \infty$. Hence $L_{-k-2} = 0$, and the series is summable $(C-\overline{k+2})$, But it is not summable $(C-\overline{k+1})$: L_{-k-1} is "properly infinite." It is easily seen that L_{-k-m} is 0 if m is a positive integer greater than unity, and is infinity if m is ositive but not integral.

8. We shall now prove the following very general theorem :—

If Σu_n is summable (Cr), where $r > -1$, then it is also summable (Cr'), where $r' > r$ (r and r' need not be integral); and the two sums are the same.

Cesàro, in the memoir already cited, stated, without proof, the following theorem :—

If $a_n/n^r \to a$, and $b_n/n^s \to b$, as $n \to \infty$, then

$$\lim_{n\to\infty} \frac{a_1 b_n + a_2 b_{n-1} + \ldots + a_{n-1} b_2 + a_n b_1}{n^{r+s+1}} = \frac{\Gamma(r+1)\,\Gamma(s+1)}{\Gamma(r+s+2)}\, ab.$$

He stated no limitations on r and s, but it appears at once from the proof (which is easy, and over which we shall not delay here) that the theorem is necessarily true only if $r > -1$, $s > -1$.

By (5), we have
$$\begin{aligned} \Sigma S_n^{(r+s)} x^n &= (1-x)^{-(r+s+1)} \Sigma u_n x^n \\ &= (1-x)^{-s} \Sigma S_n^{(r)} x^n \\ &= \Sigma A_n^{(s-1)} x^n . \Sigma S_n^{(r)} x^n. \end{aligned}$$

Hence
$$\frac{S_n^{(r+s)}}{A_n^{(r+s)}} = \frac{S_n^{(r)} A_0^{(s-1)} + S_{n-1}^{(r)} A_1^{(s-1)} + \ldots + S_1^{(r)} A_{n-1}^{(s-1)} + S_0^{(r)} A_n^{(s-1)}}{A_n^{(r+s)}}.$$

Remembering that

$$\lim_{n\to\infty} \frac{A_n^{(s-1)}}{n^{s-1}} = \frac{1}{\Gamma(s)},$$

$$\lim_{n\to\infty} \frac{S_n^{(n)}}{n^{(r)}} = \lim_{n\to\infty} \frac{S_n^{(r)}}{A_n^{(r)}} \frac{A_n^{(r)}}{n^r} = \frac{L_r}{\Gamma(1+r)},$$

$$\lim_{n\to\infty} \frac{A_n^{(r+s)}}{n^{r+s}} = \frac{1}{\Gamma(r+s+1)},$$

by an obvious application of the above theorem of Cesàro we see that

$$\begin{aligned} \lim_{n\to\infty} \frac{S_n^{(r+s)}}{A_n^{(r+s)}} &= \frac{\Gamma(s)\,\Gamma(r+1)}{\Gamma(r+s+1)} \frac{\Gamma(r+s+1)}{\Gamma(s)} \frac{L_r}{\Gamma(r+1)} \\ &= L_r, \end{aligned}$$

provided that $r > -1$, $s > 0$. Putting $r+s = r'$, we get the result as stated.

COROLLARY.—*Every series summable* (Cr) *where* $r < 0$, *and for which* $\Sigma u_n x^n$ *has a finite limit as* $x \to 1$, *is convergent.*

For, by repeated application of the theorem of § 5, we can shew that the series is summable (Ck) where $-1 < k < 0$; by the above theorem the series is then summable $(C0)$, *i.e.*, it is convergent.

Hence no oscillatory series summable (Cr), where $r > 0$, can also be summable (Cr) for $r < 0$, because $\Sigma u_n x^n$ has a finite limit as $x \to 1$, for such series. Thus abnormal series like that in § 7 differ fundamentally from those summable by positive means, in that for the former series the members of the sequence $L_{r+1}, L_{r+2}, \ldots, L_{r+m}, \ldots$ must cease to be finite before $r + m > -1$.

9. When positive integral values of r are alone considered, it is natural to speak* of the "degree of indeterminacy" k of a series, when k is the *least* positive integral value of r for which Σu_n is summable (Cr). But when r may range over the whole continuum of real number, there is not necessarily any *least* value.

The series $1-1+1-\ldots$ is an example; it is summable (Cr) provided $r > 0$, while it evidently is not summable $(C0)$.

If the series Σu_n is summable (Cr) for $r \geqslant k$, and if l is the lower limit of all such possible values of k, l will be called the *index of summability* of Σu_n; it will further be said to be *attained* or *unattained* according as Σu_n is or is not summable (Cl).

This definition is so framed to exclude such series as that of § 7, being said to have $-\infty$ as index; arbitrarily large negative values of r exist for which that series is summable (Cr); but a series will only be said to have $-\infty$ as its index when (like the geometrical series $1 + t + t^2 + \ldots$, where $|t| < 1$) it is summable (Cr) for *all* positive and negative values of r.

10. The extension to general real values of r, of Cesàro's multiplication theorem, will now be given.

THEOREM.—*If* Σu_n *is summable* (Cr), *and has sum* L_r, *and* $\Sigma u'_n$ *is summable* (Cr') *and has sum* $L'_{r'}$, *then* Σw_n *is summable* $(C\,\overline{r+r'+1})$, *and has sum* $L_r L'_{r'}$, *provided that* $r > -1$, $r' > -1$; *where*

$$w_n = u_0 u'_n + u_1 u'_{n-1} + \ldots + u_n u'_0.$$

By (5), we have $(1-x)^{-(r+1)} \Sigma u_n x^n = \Sigma S_n^{(n)} x^n$,

$$(1-x)^{-(r'+1)} \Sigma u'_n x^n = \Sigma S_n'^{(n')} x^n.$$

* As does Dr. Bromwich in his treatise already cited.

Hence $$(1-x)^{-(r+r'+2)}\Sigma w_n x^n = \Sigma S_n^{(r)} x^n . \Sigma S_n'^{(r')} x^n$$
$$\equiv \Sigma S_n''^{(r+r'+1)} x^n,$$

where $$S_n''^{(r+r'+1)} = S_0^{(r)} S_n'^{(r')} + S_1^{(r)} S_{n-1}'^{(r')} + \ldots + S_n^{(r)} S_0'^{(r')}.$$

Hence, by Cesàro's limit theorem, quoted in § 8, we have

$$\lim_{n\to\infty} \frac{S_n''^{(r+r'+1)}}{A_n^{(n+r'+1)}} = \lim_{n\to\infty} \frac{S_n^{(r)}}{A_n^{(n)}} \lim \frac{S_n'^{(r')}}{A_n^{(r')}}$$
$$= L_r L'_{r'},$$

provided that $r > -1$, $r' > -1$.

Thus the index of Σw_n does not exceed by more than unity the sum of the indices of Σu_n and $\Sigma u'_n$.

This theorem only gives an upper limit to the index of Σw_n; the real value, as will be seen in Part III of this paper, often falls below this upper limit.

The index of $(1-1+1-\ldots)^2 = 1-2+3-\ldots$ does not exceed 1, by the above theorem. It is easily seen to be 1 unattained, so that in this case the value equals the upper limit. The usual form of the multiplication theorem, where only positive integral values of r are considered, would give the upper limit of the degree of indeterminacy as 3, while the *value* of the "degree" is 2. The use of the index of a series, combined with the above generalized theorem, will usually furnish much closer information about a product series than the older method did. (*I. S.*, § 126, Ex. 2.)

Again, if $a_0 - a_1 + a_2 - \ldots$ is a convergent series with sum s (*i.e.*, index $\not> 0$), then

$$a_0 - (a_0 + a_1) + (a_0 + a_1 + a_2) - \ldots$$

has index $\not> 0$, and sum $\frac{1}{2}S$; for it is the product of $(1-1+1-\ldots)$ and $(a_0 - a_1 + a_2 - \ldots)$.

11. We shall now prove the generalization of an important theorem which, for the case when r is integral, is given in Dr. Bromwich's *Infinite Series*, § 127.

THEOREM.—*If* Σu_n *is summable* (Cr), *then* $\lim_{n\to\infty} u_n/n^r = 0$, *provided* $r > -1$.

Since, by (6), $$1 = (1-x)^{r+1}\Sigma A_n^{(r)} x^n,$$

we have, for $n > 1$,

$$0 = A_n^{(r)} - \binom{r+1}{1} A_{n-1}^{(r)} + \binom{r+1}{2} A_{n-2}^{(r)} - \ldots .$$

Making similar use of (5), we have

$$u_n = [S_n^{(r)} - L_n A_n^{(r)}] - \binom{r+1}{1} [S_{n-1}^{(r)} - L_r A_{n-1}^{(r)}] + \ldots$$
$$= A_n^{(r)} \epsilon_n + A_{n-1}^{(r)} \binom{r+1}{1} \epsilon_{n-1} + A_{n-2}^{(r)} \binom{r+1}{2} \epsilon_{n-2} + \ldots,$$

where $\epsilon_n \to 0$ as $n \to \infty$.

Remembering the properties of $A_n^{(r)}$ noticed in § 3, it is evident that

$$\frac{u_n}{n^r} = \delta_n + \binom{r+1}{1}\left(1-\frac{1}{n}\right)^r \delta_{n-1} + \binom{r+1}{2}\left(1-\frac{2}{n}\right)^r \delta_{n-2} + \ldots,$$

where $\delta_n \to 0$ as $n \to \infty$.

[*Note.*—The $(n+1)$-th term does not vanish, as the above formula would seem to indicate, but it is easily seen to tend to 0 as $n \to \infty$, provided $r > -1$. For brevity, therefore, it is here neglected, only the first n terms being considered.]

An arbitrarily small positive number ϵ being assigned, an integer m can be determined such that for all values of $n > 2m$, h_m the upper limit of

$$|\delta_n|, \ |\delta_{n-1}|, \ \ldots, \ |\delta_{n-m}|,$$

is $< \epsilon/3\kappa$, where

$$\kappa = 1 + \left|\binom{r+1}{1}\right| + \left|\binom{r+1}{2}\right| + \ldots + \left|\binom{r+1}{m}\right|.$$

Further, $$\left(1-\frac{p}{n}\right)^r < \frac{1}{2^r} \quad \text{for} \quad m \leqslant p < \frac{n}{2},$$

while $\Sigma\left|\binom{r+1}{p}\right|$ is convergent for $r > -1$, because

$$\left|\binom{r+1}{p}\right| = |A_p^{(-r-2)}|,$$

and $$\sum_{p=1}^{\infty} |A_p^{(-r-2)}| < K \sum_{p=1}^{\infty} \frac{1}{p^{r+2}}.$$

Thus, if $r > -1$, $\Sigma\left|\binom{r+1}{p}\right|$ converges with the series $\Sigma \frac{1}{p^{r+2}}$. Let R_m be its remainder after m terms. Then*

$$\sum_{p=m}^{[\frac{1}{2}n]} \left|\binom{r+1}{p}\right| \left(1-\frac{p}{n}\right)^r < \frac{R_m}{2^r}.$$

If K' be the upper limit of $|\delta_1|, |\delta_2|, \ldots$, m can be chosen large enough to satisfy the former condition and also

$$\frac{K' R_m}{2^r} < \frac{\epsilon}{3}.$$

* $[\frac{1}{2}n]$ means $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$, according as n is even or odd.

Lastly,* $\sum_{p=[\frac{1}{2}n]}^{n} \left|\binom{r+1}{p}\right| \left(1-\frac{p}{n}\right)^r |\delta_{n-p}| < K'' \sum_{[\frac{1}{2}n]}^{n} \frac{\left(1-\frac{p}{n}\right)^r}{p^{r+2}}.$

Here r may be negative, so that $\left(1-\frac{p}{n}\right)^r$ need not have any upper limit if $n-p$ remains finite. But the last expression evidently is less than

$$\frac{K'''}{n^{r+1}} \int_{\frac{1}{2}}^{1} \frac{(1-x)^r\,dx}{x^{r+2}},$$

where K''' is a constant. Since $r > -1$, n_0 can be found so that for $n > n_0$ the value of the last expression is less than $\frac{1}{3}\epsilon$.

Collecting these results, it is evident that by taking the expression for $\frac{u_n}{n^r}$ in three parts

$$\left\{\sum_{p=1}^{m} + \sum_{p=m}^{[\frac{1}{2}n]} + \sum_{p=[\frac{1}{2}n]}^{n}\right\} |\delta_{n-p}| \left(1-\frac{p}{n}\right)^r \left|\binom{r+1}{p}\right|,$$

it can (by taking $n > n_0 > 2m$) be made less than ϵ. Thus under the sole condition $r > -1$ the theorem holds as stated.

We can now give a simple example of a series which has its index attained. The series $\sum_{n=1}^{\infty} (-1)^{n+1} (\log n+1)^{-\alpha}$ is convergent if $\alpha > 0$. But since for no negative value of r does $\frac{u_n}{n^r} \to 0$, the series is only summable for values of $r \geqslant 0$, *i.e.*, its index is 0 attained.

PART II.

Oscillatory Series ($r > 0$).

§§ 12. Introductory.
13–15. Convergence factors.
16. An extension of Abel's theorem.
17. The convergence factors n^{-s}.
18. The index of $\Sigma u_n/(n+1)$ does not exceed that of Σu_n, less 1.
20. The index of a Fourier's series and its derivative.
23. The index of the series $1^s - 2^s + 3^s - \ldots$.

12. As already stated, since all series summable (Cr), where $r > 0$, are such that $\Sigma u_n x^n$ has a finite limit as $x \to 1$, no such series can be strictly divergent. Hence only finitely or infinitely oscillatory series will be considered in this section of the paper.

Neither finitely oscillatory series, nor infinitely oscillatory series, are necessarily summable

* K'' is a finite constant depending on K'.

(Cr) for any positive value of r. As examples we may quote $1-t+t^2-\ldots$ $(t>1)$ among series with infinite oscillation, and among finitely oscillatory series, the one for which

$$\Sigma u_n x^n \equiv \Sigma (-1)^n x^{a^n}.$$

Mr. Hardy has shewn* that for this series $\Sigma u_n x^n$ has no definite limit, and therefore Σu_n as so defined cannot be summable (Cr) for $r > 0$.

Another interesting possibility has been shewn to exist, by Mr. Hardy, viz., that series Σu_n may be found with as large an index as we please, while $u_n \to 0$ and $\lim_{x\to 1} \Sigma u_n x^n$ exists; thus the series $\Sigma (e^{in^b}/n^a)$, where $0 < b < 1$, $0 < a < 1$, is summable (Ck) if $a + (k+1)\, b > 1$.

Mr. Littlewood† has constructed an example of a series Σu_n for which $u_n \to 0$ and $\lim_{x\to 1} \Sigma u_n x^n$ exists, which is not summable for *any* value of r; his series is $\Sigma (e^{i (\log n)^2}/n^a)$.

13. A very considerable amount of work has recently been published on extensions, in various directions, of Abel's theorem on the limiting value of $\Sigma u_n x^n$ as $x \to 1$. Those extensions which make use of the property of Cesàro-summability generally take the form of the statement of conditions under which an oscillatory series is reduced to convergence when its terms are multiplied by corresponding members of a sequence of variable quantities (hence called by Dr. C. N. Moore *convergence factors*); and further, the additional restrictions to be placed on these convergence factors in order that, as their independent variable tends to some specified limit, the sum of the convergent series obtained by their aid may tend to the Cesàro-sum of the original series. The latest results are due to Mr. Hardy and Dr. Bromwich. See references, 2–10, § 2.

By the consideration of non-integral means, it is possible to place less stringent conditions on the convergence factors, just as we were enabled by the same means to gain more precise information concerning the index of the product of two summable series. Hence as a contribution to the abstract theory of summable oscillatory series such an extension of the known theorems seems to possess some interest.

The complete theorem falls naturally into two parts, A and B.

14. THEOREM A.—*If* Σu_n *is summable* (Cr), *or finite* (Cr), *where* $r > 0$, *then* $\Sigma u_n v_n$ *converges, provided that*

$$(a) \quad \lim_{n\to\infty} n^r v_n = 0,$$

and

$$(b) \quad \Sigma n^r \,|\, \Delta^{r+1} v_n \,| < K\,;\ddagger$$

* *Quarterly Journal of Mathematics*, Vol. 38, p. 269, 1907.

† J. E. Littlewood, "The Converse of Abel's Theorem on Power Series," *Proc. London Math. Soc.* (present volume).

‡ This, of course, implies the absolute convergence of $\Sigma n\, \Delta^{r+1} v_n$.

also its sum equals that of the series $\Sigma S_n^{(r)} \Delta^{r+1} v_n$, *which is absolutely convergent.*

Here $$\Delta^{r+1} v_n \equiv \lim_{p \to \infty} \Delta_p^{r+1} v_n,$$

provided this limit exists ; where

$$\Delta_p^{r+1} v_n \equiv v_n - \binom{r+1}{1} v_{n+1} + \binom{r+1}{2} v_{n+2} - \ldots,$$

to $p-n+1$ terms.

If condition (*a*) is satisfied, then $\Delta^{r+1} v_n$ exists. For the series which defines it is absolutely convergent when all the v's are replaced by unity, when $r > -1$; and for $r > 0$ we have $| v_n | < K$ for all values of n.

The following lemma is first required :—

LEMMA.—*Conditions* (*a*) *and* (*b*) *being satisfied, and* r *being* > 0, *we have*

$$\lim_{p \to \infty} \sum_{n-1}^{p} n^r \Delta_p^{r+1} v_n = \sum_{n=1}^{\infty} n^r \Delta^{r+1} v_n.$$

By virtue of (*a*), we can choose p_0, an arbitrarily small positive number ϵ having been assigned, such that for $m > p_0$,

$$| v_m m^r | < \epsilon ;$$

also we have $$\binom{r+1}{m} < \frac{K}{m^{r+2}},$$

where K is a constant for all values of m.

Hence* if $p > p_0$,

$$n^r | \Delta^{r+1} v_n - \Delta_p^{r+1} v_n | = n^r \left| v_{p+1} \binom{r+1}{p-n+1} - v_{p+2} \binom{r+1}{p-n+2} + \cdots \right|$$

$$< K\epsilon \left(\frac{n}{p}\right)^r \left\{ \frac{1}{(p-n+1)^{r+2}} + \frac{1}{(p-n+2)^r} + \cdots \right\}$$

$$< K\epsilon \frac{1}{(p-n+1)^{r+1}},$$

by an easy integral summation, provided $n < p$.

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* K is used here, and throughout the subsequent work, in the sense introduced by Borel.

Hence $$\left| \sum_{n=1}^{p} n^r (\Delta_p^{r+1} v_n - \Delta^{r+1} v_n) \right| < K\epsilon \sum_{n=1}^{p} \frac{1}{(p-n+1)^{r+1}}$$

$$< K\epsilon \sum_{1}^{\infty} \frac{1}{n^{r+1}} < K\epsilon,$$

where K is independent of p. The truth of the lemma is an immediate consequence.

15. To prove the main proposition we proceed thus :—

By a transformation which is easy, but tedious to write out, it follows that

$$\sum_{n=0}^{p} u_n v_n = \sum_{n=0}^{p} S_n^{(r)} \Delta_p^{r+1} v_n,$$

by making use of the identity (7).*

Now, provided that Σu_n is summable or finite (Cr), we have

$$\left| \sum_{n=0}^{p} S_n^{(r)} (\Delta_p^{r+1} v_n - \Delta^{r+1} v_n) \right| < K \left| \sum_{n=0}^{p} n^r (\Delta_p^{r+1} v_n - \Delta^{r+1} v_n) \right| ;$$

K is a constant not depending on p. Hence, by the lemma just established,

$$\lim_{p \to \infty} \sum_{n=0}^{p} S_n^{(r)} \Delta_p^{r+1} v_n = \sum_{n=0}^{\infty} S_n^{(r)} \Delta^{r+1} v_n,$$

which is absolutely convergent by comparison with the series

$$\Sigma n^r \, | \, \Delta^{r+1} v_n \, | \, .$$

Therefore $$\sum_{0}^{\infty} u_n v_n = \sum_{0}^{\infty} S_n^{(r)} \Delta^{r+1} v_n,$$

as the theorem states.

16. THEOREM B.—*If the convergence factors v_n are functions of a variable x, and are such that $\lim_{x \to c} v_n = 1$, then, provided that conditions (a) and (b) hold for all values of $x > c$, the K of condition (b) being a constant, we have*

$$\lim_{x \to c} \Sigma u_n v_n = L_r.$$

We first notice that $\lim_{x \to c} \Delta^{r+1} v_n = 0$ for all values of n; for $\Delta^{r+1} v_n$ is

* A similar transformation is set out at length in Dr. Bromwich's paper in *Math. Annalen*, Bd. 65.

an absolutely convergent series, and so also is

$$S' = 1-\binom{r+1}{1}+\binom{r+1}{2}-\ldots = (1-1)^{r+1} = 0.$$

Since $|v_n| < K$ for all values of $x > c$, and all values of n, $\Delta^{r+1}v_n$ is uniformly convergent for $x > c$, by Weierstrass's M-test, S' being the comparison series. Hence

$$\lim_{x\to c} \Delta^{r+1}v_n = S' = 0.$$

Using a device which is frequently of service in this kind of work, we consider the convergent series for which

$$a_0 = 1, \quad a_2 = a_3 = \ldots = 0, \quad s_0' = s_1' = \ldots = 1, \quad S_n'^{(r)} = A_n^{(r)}.$$

Thus, by Theorem A,

$$\sum_0^\infty a_n v_n = v_0 = \sum_0^\infty S_n'^{(r)} \Delta^{r+1} v_n = \sum_0^\infty A_n^{(r)} \Delta^{r+1} v_n.$$

Hence
$$(\Sigma u_n v_n - L_r v_0) = \sum_0^\infty [S_n^{(r)} - L_r A_n^{(r)}] \Delta^{r+1} v_n,$$

and since
$$\frac{S_n^{(r)}}{A_n^{(r)}} \to L_r \quad \text{as} \quad n \to \infty,$$

we have
$$|S_n^{(r)} - L_r A_n^{(r)}| < \epsilon_{n_0} n^r \quad \text{for} \quad n > n_0;$$

ϵ_{n_0} can, of course, be taken as small as we please by taking n_0 sufficiently large.

Consequently,

$$\left| \sum_0^\infty (S_n^{(r)} - L_r A_n^{(r)}) \Delta^{r+1} v_n \right| < \left| \sum_0^{n_0} (S_n^{(r)} - A_n^{(r)} L_r) \Delta^{r+1} v_n \right| + \epsilon_{n_0} \left| \sum_{n_0}^\infty n^r \Delta^{r+1} v_n \right|$$

$$< \delta + \epsilon_{n_0} \sum_0^\infty n^r \Delta^{r+1} v_n$$

$$< \delta + K\epsilon_{n_0},$$

where δ can be taken as small as we please by choosing a sufficiently small interval for $x-c$, since

$$\lim_{x\to c} \Delta^{r+1} v_n = 0,$$

and n_0 is fixed.*

* $\Delta^{r+1}v_n$ does not necessarily $\to 0$, as $x \to c$, uniformly for all values of n.

Thus $$\lim_{x\to c} \sum_{0}^{\infty} (S_n^{(r)} - A_n^{(r)} L_r) \Delta^{r+1} v_n = 0,$$

whence it follows that

$$\lim_{x\to c} \Sigma u_n v_n = L_r \lim_{x\to c} v_0 = L_n.$$

17. After $v_n = x^n$, $c = 1$, which evidently satisfy the conditions of Theorems A and B, and lead to part of the general theorem of § 6, the most important convergence factors are $v_n = (n+1)^s$. Consequently we shall prove that they satisfy the general conditions, and so deduce the following theorem* :—

If Σu_n is summable or finite (Cr) $r > 0$, then $\Sigma u_n (n+1)^{-s}$ is convergent, provided $s > r$.

Condition (*a*) is obviously satisfied. It only remains to shew that

$$\Sigma n^r \, | \Delta^{r+1} (n+1)^{-s} |$$

is convergent.

We have $$\Gamma(s) m^{-s} = \int_0^\infty e^{-mx} x^{s-1} \, dx.$$

Hence $$\Gamma(s) \Delta^{r+1} v_n = \sum_{m=0}^{\infty} \left[\int_0^\infty e^{-(m+n+1)x} x^{s-1} \, dx \, (-1)^m \binom{r+1}{m} \right].$$

By a well known test,† the order of summation and integration can be inverted, and we thus obtain

$$\Gamma(s) \Delta^{r+1} v_n = \int_0^\infty e^{-(n+1)x} x^{s-1} \left[\sum_{m=0}^{\infty} (-1)^m \binom{r+1}{m} e^{-mx} \right] dx$$

$$= \int_0^\infty e^{-(n+1)x} x^{s-1} (1 - e^{-x})^{r+1} \, dx.$$

Thus $\Delta^{r+1} n^{-s}$ is positive, and decreases as n increases, just as when r is integral.

None of the ordinary tests seem to suffice to prove that

$$\Sigma n^r \, \Delta^{r+1} (n+1)^{-s}$$

is absolutely convergent ; we therefore proceed as follows. Consider the integral

$$I_\epsilon \equiv \int_\epsilon^\infty x^{s-1} (1 - e^{-x})^{r+1} \sum_{n=m}^{p+m} n^r e^{-(n+1)x} \, dx.$$

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* See the note to § 2 of the Introduction, p. 371.

† See, for example, Bromwich's *Infinite Series*, § 176 A.

A finite constant K_m can be found such that the n-th term of the series $\sum_{n=0}^{p}(n+m)^r e^{-nx}$ is less than the corresponding term of the series

$$K_m \sum_{n=0}^{\infty} A_n^{(r)} e^{-nx} = \frac{K_m}{(1-e^{-x})^{r+1}} \quad (x \geqslant \epsilon > 0)$$

for all values of n, m being fixed. Here K_m is independent of ϵ, but is a function of m. Evidently

$$K_m < K'(m+1)^r,$$

where K' is independent of both m and ϵ. Therefore

$$\sum_{n=m}^{m+p} n^r e^{-(n+1)x} \leqslant \frac{K_m e^{-(m+1)x}}{(1-e^{-x})^{r+1}}$$

for all values of p. Consequently

$$I_\epsilon < K_m \int_\epsilon^\infty x^{s-1} e^{-(m+1)x}\, dx$$

or

$$\sum_{n=m}^{m+p} n^r \int_\epsilon^\infty x^{s-1}(1-e^{-x})^{r+1} e^{-(n+1)x}\, dx < \frac{K_m \Gamma(s)}{(m+1)^s} < \frac{K' \Gamma(s)}{(m+1)^{s-r}}$$

for all values of p and all positive values of ϵ (< 1, say). Now let $\epsilon \to 0$. Since the number of terms on the left is finite, we have

$$\sum_{n=m}^{m+p} n^r \lim_{\epsilon \to 0} \int_\epsilon^\infty x^{s-1}(1-e^{-x})^{r+1} e^{-(n+1)x}\, dx = \sum_{n=m}^{m+p} n^r \int_0^\infty x^{s-1}(1-e^{-x})^{r+1} e^{-(n+1)x}\, dx$$

$$= \Gamma(s) \sum_{n=m}^{m+p} n^r \Delta^{r+1} v_n \leqslant \frac{K' \Gamma(s)}{(m+1)^{s-r}}.$$

Hence, since K' is independent of p, and $s > r$, the series $\Sigma n^r \Delta^{r+1} v_n$ is absolutely convergent.

The theorem above has an immediate corollary: *If Σu_n is summable by Cesàro's method and has an index of summability k (> 0), then $\sum_0^\infty u_n (n+1)^{-l}$ is convergent, provided $l > k$.*

Obviously this corollary enables us to obtain less rapidly diminishing convergence factors of the type n^{-s} than the theorems dealing only with positive integral orders of summation; and the analysis by which the result has been obtained is only very slightly more complex than that needed for those theorems.

The corollary gives us some information about the index of a series (supposed summable). If k is the lower limit of the positive numbers l, for which $\Sigma u_n n^{-l}$ is convergent, the index of Σu_n is $\geqslant k$. If Σu_n is summable of order r, then $u_n/n^r \to 0$ as $u \to \infty$,; hence $\Sigma u_n/n^{r+2}$ is absolutely convergent. Therefore such values of l always exist and have a lower limit. But the series may be reducible to convergence by such factors and yet not be summable, of course.

If the index does exist, however, we can find a lower limit to its possible values. Thus $1^s-2^s+3^s\ldots$ cannot have an index $< s$ $(s > 0)$. Also, since $u_n/n^s = \pm 1$, if the index is s, it is unattained.

Many further types of convergence factors might be investigated, but I shall not here multiply examples of such, as the general consequences of the extension of the theorems to non-integral values of r have been sufficiently indicated.

18. Dr. H. Bohr* and Dr. Marcel Riesz† have given the theorem that, if Σu_n is summable (Cs), then $\Sigma u_n/(n+1)$ is summable $(C\overline{s-1})$, s being a positive integer. The following extension‡ of this theorem is interesting in itself, and will be of use later.

THEOREM.—*If* $\sum_0^\infty u_n$ *is summable or finite* (Ck) *whether* k *is integral or not,* POSITIVE OR NEGATIVE, *then* $\Sigma u_n/(n+1)$ *is summable* $(C\overline{k+1})$.

Let T_n be the function related to the series $\Sigma u_n/(n+1)$ in the same way as $S_n^{(k-1)}$ is related to Σu_n. Then, by (2) and (3), we have

$$T_n = \sum_{r=0}^{n} A_{n-r}^{(k-1)} \frac{u_r}{n+1} = \sum_{r=0}^{n} S_r^{(k)} B_{n,r},$$

substituting for u_r from (7) and making a transformation similar to that in § 15. Here $B_{n,r}$ denotes

$$\frac{A_{n-r}^{(k-1)}}{r+1} - \binom{k+1}{1} \frac{A_{n-r-1}^{(k-1)}}{r+2} + \binom{k+1}{2} \frac{A_{n-r-2}^{(k-1)}}{r+3} - \ldots$$

to $n-r+1$ terms (as the theorem has previously been proved for the case when k is a positive integer, and, as k cannot be 0 or a negative integer, it will be supposed that k is non-integral). Evidently

$$\sum_{n=r}^{\infty} B_{n,r} x^{n+1} = (1-x)^{-k} \left\{ \frac{x^{r+1}}{r+1} - \binom{k+1}{1} \frac{x^{r+2}}{r+2} + \ldots \right\}$$

$$= (1-x)^{-k} \int_0^x x^r (1-x)^{k+1} dx \quad (x < 1). \qquad (\alpha)$$

* *Comptes Rendus*, January 11th, 1909.

† *Ibid.*, June 21st, 1909. These theorems were not stated as above, and also included further results which are not considered here; but the form above is immediately deducible from the theorems.

‡ See the note to § 2 of the Introduction, p. 371.

Integrate $(r-1)$ times by parts. We get

$$\sum_{n=r}^{\infty} B_{n,r}x^{n+1}$$

$$= \frac{1}{(1-x)^k}\left\{-\frac{x^r(1-x)^{k+2}}{k+2} - \frac{r}{k+2}\frac{x^{r-1}}{k+3}(1-x)^{k+3} - \ldots\right.$$

$$\left. - \frac{r!\,x(1-x)^{k+r+1}}{(k+2)(k+3)\ldots(k+r+1)} + \frac{r!}{(k+2)\ldots(k+r+1)}\int_0^x (1-x)^{k+r}dx\right\}$$

$$= \frac{1}{(1-x)^k}\left\{-\frac{x^r}{k+2}(1-x)^{k+3} - \ldots - \frac{r!\,(k+1)!}{(k+r+2)!}\left[(1-x)^{k+r+2}-1\right]\right\}. \quad (\beta)$$

The coefficients of x^n on the right vanish, as we know from (α), up to $n = r+1$; also it is easy to see that the coefficient of x^{r+1} is $(r+1)^{-1}$, and of x^{r+2} is $(k-r-1)(r+1)^{-1}(r+2)^{-1}$. The coefficients of the higher terms come, as is seen from (β), from the expansion of the term

$$\frac{r!\,(k+1)!}{(k+r+2)!}\frac{1}{(1-x)^k} = \frac{r!\,(k+1)!}{(k+r+2)!}\sum_0^{\infty} A_n^{(k-1)}x^n.$$

Hence

$$B_{n,n} = \frac{1}{n+1}, \qquad B_{n,n-1} = \frac{k-n}{n(n+1)}, \qquad B_{n,r} = \frac{r!\,(k+1)!}{(k+r+2)!}A_{n+1}^{(k-1)},$$

for $r \leqslant n-2$. Therefore

$$\frac{T_n}{A_n^{(k-1)}} = \frac{A_{n+1}^{(k-1)}}{A_n^{(k-1)}}\sum_{r=0}^{n-2}\frac{r!\,(k+1)!}{(k+r+2)!}S_r^{(k)} + \frac{k-n}{n(n+1)}\frac{S_{n-1}^{(k)}}{A_n^{(k-1)}} + \frac{1}{n+1}\frac{S_n^{(k)}}{A_n^{(k-1)}}.$$

The *sum* of the last two terms is readily seen to tend to 0, though they separately do not $\to 0$ as $n \to \infty$. Also

$$\frac{r!\,(k+1)!}{(k+r+2)!} = \frac{k+1}{(k+r+1)(k+r+2)}\frac{1}{A_r^{(k)}}.$$

Hence $$\frac{T_n}{A_n^{(k-1)}} = (k+1)\frac{A_{n+1}^{(k-1)}}{A_n^{(k-1)}}\sum_{r=0}^{n-2}\frac{1}{(k+r+1)(k+r+2)}\frac{S_r^{(k)}}{A_r^{(k)}} + \epsilon_n,$$

where $\epsilon_n \to 0$ as $n \to \infty$.

Since $|S_r^{(k)}/A_r^{(k)}| < K$ for all value of r, by comparison with the series $\Sigma 1/r^2$, we see that

$$\lim_{n\to\infty}\frac{T_n}{A_n^{(k-1)}} = (k+1)\sum_{r=0}^{\infty}\frac{1}{(k+r+1)(k+r+2)}\frac{S_r^{(k)}}{A_r^{(k)}},$$

the latter series being absolutely convergent. This proves the theorem.

We thus have as a corollary:—*The index of a series* $\sum_0^{\infty} u_n/(n+1)$ *does not exceed* $k-1$, *where* k *is the index of* Σu_n.

By repeated application of the theorem, we have: *The index of a series* Σu_n *being* k, *the index of* $\Sigma u_n/(n+1)^m$ *is* $\geqslant k-m$, m *being a positive integer* ($k-m$ *must not be a negative integer*).

As applied to convergent (or finitely oscillatory) series, the theorem must thus be stated: Σu_n *being a convergent or finitely oscillatory series* [such are easily seen to be always finite (Cr) for any positive value of r], *the series* $\Sigma\{u_n/(n+1)\}$ *is summable* (Cr) *for any value of* $r > -1$, *i.e., its index* (whose existence is thus proved) *is* $\geqslant -1$.

For the series is finite $(C\epsilon)$ where ϵ is any positive number, however small; hence $\Sigma u_n/(n+1)$ is summable $(C\,\overline{\epsilon-1})$.

19. One of the most interesting of the applications of Cesàro's method was that made by Fejér in a number of papers* on Fourier's series. He shewed that, if $f(x)$ is integrable and periodic in 2π, then the Fourier's series

$$\sum_0^\infty (a_n \cos x + b_n \sin nx)$$

where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha)\,d\alpha,$$

$$\begin{matrix} a_n \\ b_n \end{matrix} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha) \begin{matrix} \cos \\ \sin \end{matrix} n\alpha\, d\alpha$$

is summable $(C1)$, and has the sum† $\frac{1}{2}[f(x+0)+f(x-0)]$ at any point x at which the latter limit is definite.

Fejér considered only the case when $f(x)$ has a Riemann integral; wide generalizations of his theorems have been published by Dr. Hobson and Lebesgue, in which $f(x)$ is subject to the less stringent condition of Lebesgue-integrability.‡

These theorems may still further be extended§ by the consideration of non-integral orders of summability, and we so obtain the following theorem, which is of some theoretical interest:—

If $f(x)$ *be any function, limited or unlimited, in* $(-\pi, \pi)$, *which has a Lebesgue integral, and therefore a corresponding Fourier series, then*

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* See memoirs 11–13 of the list appended to § 2.

† This expression, of course, denotes $\lim_{h\to 0} \frac{1}{2}[f(x+h)-f(x-h)]$.

‡ For references to original memoirs and a full account of the whole subject, reference may be made to Hobson's *Theory of Functions of a Real Variable*, ch. vii.

§ Only the main outline of the proof will be given; it is similar in many ways to the proof in §§ 469 *et seq.* of Dr. Hobson's treatise, where the series is shewn to be summable, and to this place reference may be made to complete the detailed proof.

the latter series is summable (Ck) *for any value of* $k > 0$, *and the sum is equal to* $\frac{1}{2}[f(x+0)+f(x-0)]$ *at any point* x *at which this limit exists. Further, the convergence as* $n \to \infty$ *of the sum-function* $S_n^{(k)}/A_n^{(k)}$ *is uniform in any interval contained within an interval in which* $f(x)$ *is limited and at every point of which* $f(x+0)+f(x-0)$ *exists.*

Thus at such points, if a Fourier's series is not convergent, it only just oversteps the bounds of convergence. Its index of summability is then zero.

20. We shall suppose, for convenience, that $0 < k < 1$; this, of course, implies no ultimate restriction. The point x is understood to be one at which the limit $[f(x+0)+f(x-0)]$ exists as a definite number. We have

$$s_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\alpha') \frac{\sin \frac{2n+1}{2} (\alpha'-x)}{\sin \frac{1}{2}(\alpha'-x)} d\alpha'.$$

Write $\alpha' = x+2\alpha$. Remembering that $f(x)$ is defined as a periodic function, for values of x not necessarily in the interval $(-\pi, \pi)$, we have

$$s_n(x) = \frac{1}{\pi} \int_0^{\frac{1}{2}\pi} [f(x+2\alpha)+f(x-2\alpha)] \frac{\sin (2n+1)\alpha}{\sin \alpha} d\alpha.$$

Hence, denoting $S_n^{(k)}/A_n^{(k)}$, for the point x, by $L_n(x)$, we have

$$L_n(x) = \frac{1}{\pi} \int_0^{\frac{1}{2}\pi} [f(x+2\alpha)+f(x-2\alpha)] \phi(\alpha, n) d\alpha,$$

where $$\phi(\alpha, n) = \frac{1}{A_n^{(k)}} \sum_{r=0}^{n} A_r^{(k-1)} \frac{\sin (2\overline{n-r}+1)\alpha}{\sin \alpha}.$$

Since $$\int_0^{\frac{1}{2}\pi} \frac{\sin (2n+1)\alpha}{\sin \alpha} d\alpha = \frac{\pi}{2},$$

we have $$\frac{1}{\pi} \int_0^{\frac{1}{2}\pi} \phi(\alpha, n) d\alpha = \tfrac{1}{2}.$$

Therefore $$L_n(x) - \tfrac{1}{2}[f(x+0)+f(x-0)] = \frac{1}{\pi} \int_0^{\frac{1}{2}\pi} F(\alpha, x)\, \phi(\alpha, n) d\alpha,$$

where $$F(\alpha, x) = f(x+2\alpha)+f(x-2\alpha)-f(x+0)-f(x-0),$$

so that $$\lim_{\alpha \to 0} F(\alpha, x) = 0.$$

Now, by Abel's lemma, since $0 < k < 1$,

$$\left| \sum_{r=0}^{n} A_r^{(k-1)} \sin (2n+1)\alpha \right| < K,$$

where K is a finite constant independent of n and a, provided that

$$0 < \epsilon \leqslant a \leqslant \frac{\pi}{2}.$$

Hence $$|\phi(a, n)| < \frac{K}{A_n^{(k)}}$$

for $$0 < \epsilon \leqslant a \leqslant \frac{\pi}{2}.$$

Thus we have

$$\left| \frac{1}{\pi} \int_\epsilon^{\frac{1}{2}\pi} F(a, x)\, \phi(a, n)\, da \right| < \frac{K}{A_n^{(k)}} \int_\epsilon^{\frac{1}{2}\pi} |F(a, x)|\, da,$$

whence it follows that n_0 can be so chosen that, for $n > n_0$,

$$\left| \frac{1}{\pi} \int_\epsilon^{\frac{1}{2}\pi} F(a, x)\, \phi(a, n)\, da \right| < \sigma;$$

and moreover, if x lies within an interval contained in an interval in which $f(x)$ is limited and at every point of which $f(x+0)+f(x-0)$ exists, n_0 can be so chosen, independently of x, that this inequality is satisfied for all points x in this interval.

We shall now show that

$$\frac{1}{\pi} \int_0^\epsilon |\phi(a, n)|\, da < K,$$

where K is a constant which is independent of n.*

Since for $0 < a < \pi/n$,

$$\left| \frac{\sin (2n-r+1)\, a}{\sin a} \right| < 2n,$$

it is easily seen that

$$\int_0^{\pi/n} |\phi(a, n)|\, da < \frac{2n}{A_n^{(k)}} \frac{\pi}{n} \{A_0^{(k-1)} + \ldots + A_n^{(k-1)}\} = \frac{2\pi}{A_n^{(k)}} A_n^{(k)} = 2\pi.$$

Hence we have only to shew that

$$\int_{\pi/n}^\epsilon |\phi(a, n)|\, da < K;$$

---

* K will have this meaning throughout the remainder of § 20; but its value is not necessarily the same in each inequality, or even in different parts of the same inequality.

this involves some rather tedious but not very difficult analysis, and is best done (as was kindly pointed out to me by Mr. Littlewood) by replacing the *sum* $\phi(a, n)$ by the analogous *integral* $\psi(a, n)$, where

$$\psi(a, n) = \frac{\{\Gamma(k)\}^{-1}}{A_n^{(k)} \sin a} \int_1^{n+1} x^{k-1} \sin(2n+1-2x)\, a\, dx.$$

Evidently

$$\psi(a, n) = \frac{\{\Gamma(k)\}^{-1}}{A_n^{(k)} a^k \sin a} \int_a^{(n+1)a} y^{k-1} \sin(2\overline{n+1}\, a - 2y)\, dy.$$

Since we are considering values of a ranging from π/n to ϵ, the lower limit of the last integral will vary between these values, while the upper limit will range from π to $n^{1-k}\pi$. Hence, writing $2n+1 = n'$,

$$\psi(a, n) = \frac{\{\Gamma(k)\}^{-1}}{A_n^{(k)} a^k \sin a} \left\{ \int_a^{\pi} y^{k-1} \sin(n'a - 2y)\, dy + \int_{\pi}^{(n+1)a} y^{k-1} \sin(n'a - 2y)\, dy \right\}.$$

Since $0 < k < 1$, the first of the integrals within the last bracket has a finite upper limit independent of n, and the same is true of the second, because the infinite integral

$$\int_{\pi}^{\infty} y^{k-1} \sin(n'a - 2y)\, dy$$

is convergent (though not absolutely). Consequently

$$|\psi(a, n)| < \frac{K}{A_n^{(k)} a^k \sin a} < \frac{K'}{n^k a^{k+1}};$$

and therefore $\int_{\pi/n}^{\epsilon} |\psi(a, n)|\, da < K' n^{-k} \int_{\pi/n}^{\epsilon} a^{-k-1}\, da < \frac{K n^k}{n^k} = K.$

Hence it only remains to shew that

$$\int_{\pi/n}^{\epsilon} |\phi(a, n) - \psi(a, n)|\, da < K.$$

Now, by means of Stirling's expression for the gamma function (*I.S.*, p. 462), it is not difficult to prove that for $n \geqslant 1$, $k > 0$,

$$|\Gamma(k) A_n^{(k-1)} - n^{k-1}| < A n^{k-\frac{3}{2}},$$

where A is a constant independent of n. Therefore

$$\left| \phi(\alpha, n) - \frac{\{\Gamma(k)\}^{-1}}{A_n^{(k)} \sin \alpha} \sum_{m=1}^{n} m^{k-1} \sin (2\overline{n-m}+1)\alpha \right|$$

$$< \frac{K}{A_n^{(k)} \sin \alpha} \sum_{m=1}^{n} m^{k-\frac{3}{2}} < \frac{K'}{n^k \sin \alpha} n^{k-\frac{1}{2}} < \frac{K}{\alpha n^{\frac{1}{2}}};$$

and since

$$\int_{\pi/n}^{\epsilon} \frac{K\,d\alpha}{\alpha n^{\frac{1}{2}}} < \frac{A}{n^{\frac{1}{2}}} \log n,$$

which tends to zero as $n \to \infty$, our problem is finally reduced to proving that

$$\frac{1}{A_n^{(k)}} \int_{\pi/n}^{\epsilon} \frac{d\alpha}{\sin \alpha} \left\{ \sum_{m=1}^{n} m^{k-1} \sin (2\overline{n-m}+1)\alpha - \int_1^{n+1} x^{k-1} \sin (n'-2x)\alpha\,dx \right\}$$

has a finite upper limit as $n \to \infty$.

We notice that

$$m^{k-1} \sin (n'-2m)\alpha - \int_m^{m+1} x^{k-1} \sin (n'-2x)\alpha\,dx$$

$$= m^{k-1} \sin (n'-2m)\alpha - \int_0^1 (m+y)^{k-1} \sin (n'-2m-2y)\alpha\,dy$$

$$= m^{k-1} \sin (n'-2m)\alpha \left[1 - \int_0^1 \left(1+\frac{y}{m}\right)^{k-1} \cos 2y\alpha\,dy\right]$$

$$+ \int_0^1 (m+y)^{k-1} \sin 2y\alpha\,dy \cos (n'-2m)\alpha.$$

But

$$\left(1+\frac{y}{m}\right)^{k-1} = 1+\frac{\delta}{m},$$

where δ has a finite upper limit for all values of y and m considered. Hence the above difference is equal to

$$m^{k-1} \sin (n'-2m)\alpha \left[1 - \frac{\sin 2\alpha}{2\alpha} + \frac{K}{m}\right]$$

$$- \int_0^1 (m+y)^{k-1} \sin (n'-2m)\alpha \sin 2y\alpha\,dy,$$

where K has a finite upper limit for all values of m and α under consideration.

Summing from $m=1$ to $m=n$, we have

$$\frac{1}{A_n^{(k)}}\left\{\sum_{m=1}^{n} m^{k-1}\sin(n'-2m)\,\alpha-\int_1^{n+1} x^{k-1}\sin(n'-2x)\,\alpha\,dx\right\}$$

$$=\frac{K}{n^k}\left[\sum_{m=1}^{n} m^{k-1}\cos(n'-2m)\,\alpha\left(K'\alpha^2+\frac{K''}{m}\right)\right.$$

$$\left.-\int_0^1\left\{\sum_{m=1}^{n}(m+y)^{k-1}\sin(n'-2m)\,\alpha\right\}\sin 2y\,\alpha\,dy\right],$$

K' being a constant depending on α but not on m. Now

$$\left|\sum_{m=1}^{n} m^{k-1}\cos(n'-2m)\,\alpha\right|<An^{k-1}/\sin\alpha,$$

$$\left|\sum_{m=1}^{n}(m+y)^{k-1}\sin(n'-2m)\,\alpha\right|<A'n^{k-1}/\sin\alpha,$$

$$\left|\sum_{m=1}^{n} m^{k-2}\cos(n'-2m)\,\alpha\right|<A''n^{k-1}.$$

Hence

$$\frac{1}{A_n^{(k)}}\left\{\sum_{m=1}^{n} m^{k-1}\sin(n'-2m)\,\alpha-\int_1^{n+1} x^{k-1}\sin(n'-2x)\,\alpha\,dx\right\}$$

$$<A_1\alpha+A_2+\int_0^1 A'\frac{\sin 2y\,\alpha}{\sin\alpha}\,dy<K;$$

and therefore the integral of the left-hand side, from $\alpha=\pi/n$ to $\alpha=\epsilon$, is also less than K.

This completes the proof that

$$\int_0^{\epsilon}|\phi(\alpha,n)|\,d\alpha<K,$$

where K is independent of n.

We can now see that

$$\int_0^{\epsilon} F(\alpha,x)\,\phi(\alpha,n)\,d\alpha<K\overline{F(\epsilon,x)},$$

where $\overline{F(\epsilon,x)}$ is the upper limit of $F(\alpha,x)$ in the interval $0\leqslant\alpha\leqslant\epsilon$. Now as $\epsilon\to 0$, $\overline{F(\epsilon,x)}\to 0$, and this convergence to the limit is, moreover, uniform in any interval at every point of which $f(x+0)+f(x-0)$ exists, $f(x)$ being limited in the interval. Therefore the same is true also of

$$\int_0^{\epsilon} F(\alpha,x)\,\phi(\alpha,n)\,d\alpha.$$

Collecting our results, we see that

$$\lim_{n\to\infty} L_n(x) = \tfrac{1}{2}[f(x+0)+f(x-0)],$$

and that this convergence to the limit—*i.e.*, the summability (Ck)—is uniform in such an interval as has been specified. This proves the theorem.*

21. Fejér has also shown that, if $f'(x)$ satisfy the condition of integrability, then the series obtained by term-by-term differentiation of the Fourier's series for $f(x)$ is summable $(C1)$ with a sum equal to $f'(x)$, save at $\pm\pi$, if $f(x)$ has a finite discontinuity there. It is easily proved that the differentiated series has an index $\nrightarrow 0$.

For the Fourier's series for $f'(x)$ is

$$a_0'+\Sigma(a_n'\cos nx+b_n'\sin nx),$$

where

$$a_0' = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(a)\,da = \frac{q-p}{2\pi},$$

$$a_n' = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(a)\cos na\,da = \frac{q-p}{\pi}+nb_n,$$

$$b_n' = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(a)\sin na\,da = -na_n,$$

where p, q are the limiting values of $f(x)$ at $-\pi$, π.

$$\text{Thus}\quad f'(x) = \frac{q-p}{2\pi}+\sum_{n=1}^{\infty}\left[\frac{q-p}{\pi}\cos nx+n(b_n\cos nx-a_n\sin nx)\right].$$

Now the series $\frac{q-p}{\pi}[\frac{1}{2}+\Sigma\cos nx]$ may very easily be shewn† to have index 0 and sum 0 save at the points $x=0,\ 2\pi,\ \ldots$. Hence, by subtraction it follows that

$$f'(x) = \Sigma n(b_n\cos nx-a_n\sin nx),$$

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* This theorem has been proved in an entirely different way by Dr. Riesz (*Comptes Rendus*, November 22, 1909), as was pointed out to me by Mr. Hardy.

In a paper shortly to be published in the *Quarterly Journal of Mathematics*, and entitled "Notes on the General Theory of Summability, with applications to Fourier's and other Series," I have proved the same theorem (with various others) as a particular example of a systematic theory.

† This is a very special case of the theorems of §§ 25, 26, but of course a much simpler proof can be given in this particular case.

except at 0 and 2π, if $q \neq p$, and that this series has index $\not> 0$; and this series is the one got by term-by-term differentiation of the Fourier's series for $f(x)$.

22. In Dr. Bromwich's *Infinite Series*, it is proved that the series $1^s - 2^s + 3^s - \ldots$, where s is integral, is summable $(C\ \overline{s+1})$. The method there employed in the proof may easily be adapted to shew that the series is *finite* (Cs). It therefore seems probable that the index of the series is s unattained, and that this is also true even when s is not integral. This theorem seems by no means easy to establish directly; but the following chain of argument, based on theorems already published* (without proofs) by M. Bohr and Dr. Marcel Riesz, appears to meet the case.

M. Bohr has shewn that if Σu_n is finite (Cs), s being integral, then $\Sigma u_n/n^\alpha$ is summable (Cs), if $\alpha > 0$. Next, Dr. Riesz has proved that if Σa_n is summable (Cs), s being integral, then $\Sigma a_n/n^k$ (where $k > 0$, integral or not) is summable $(C\ \overline{s-k})$. Hence $1^p - 2^p + 3^p - \ldots$, where

$$p = s - \alpha - k,$$

is summable $(C\ \overline{s-k})$. Therefore, r being any positive number, integral or not, an integer s, and numbers k and α, can be so chosen that $s - k - \alpha = r$, whence it follows that $1^r - 2^r + 3^r - \ldots$ is summable $(C\ \overline{r+\alpha})$, where α is only subject to the condition of being positive, however small. Hence the index of the series is n, and it is evidently (by § 11) unattained.

By the theorem of § 18 the same result holds good also when s is negative, the series then being convergent.

Though we are compelled to adopt the above indirect proof to establish the theorem that the index of $1^s - 2^s + 3^s - \ldots$ (s positive or negative) is s, when Cesàro's method of summation is used, it is not very difficult to prove the same result directly when, instead of Cesàro's, we use Dr. Riesz's method of summation in the particular case $\lambda(n) = n$; and, as we have already mentioned, the two methods are coextensive (§ 3).

The direct proof is given in § 26 of Part III of this paper. It applies not only to convergent series (when s is negative), but also to all values of s, integral or not, positive or negative, real or complex.

Moreover, the proof is easily modified, as is there explained, so as to apply to the case of the more general series $\Sigma n^s e^{\alpha i n}$, where $0 < \alpha < 2\pi$.

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* See (8) and (9) in the list of references.

We thus find that, for all values of s, the series

$$\Sigma n^s \cos na,$$

$$\Sigma n^s \sin na,$$

are summable (R, n, s') if $s' > s$, and $0 < a < 2\pi$.

Further, it is readily proved that for values of a such that

$$|\sin a| > \epsilon > 0,$$

ϵ being any fixed positive quantity, these series are uniformly summable with respect to a.

If we care to consider complex values of s', corresponding to complex orders of summability, the theorems still hold good provided that the real part of s' is greater than the real part of s.

PART III.

Convergent Series.

§§ 23. The index of a convergent series.
24. The index of $1^{-s}-2^{-s}+3_{-s}$.
25, 26. Direct determination of this index, for Riesz's method.
27. The "dilution" of series.
28. The index of an absolutely convergent series.

23. It seems advisable here to say a few words on the use and meaning of the index of a convergent series. Absolutely convergent series are usually regarded as being essentially more convergent than conditionally or semi-convergent series (as the very names imply). But, as we shall shew, a series may be absolutely convergent and yet have index 0 (attained, of course), so that a semi-convergent series may have a lower index than an absolutely convergent series. The index only measures the evenness and the rate of the approach of the partial sum s_n to its limit.

My original object in considering the summability of convergent series was to endeavour to prove the multiplication theorems for an absolutely convergent series with a like series or with a semi-convergent series, as particular cases of the general multiplication theorem of § 10. But it is evident that the latter theorem would then require the index of an absolutely convergent series to be -1, and § 11 shews that this is not usually the case. Thus in the case of an absolutely convergent series a knowledge of its index seems to be of little interest; in the case of semi-convergent series, however, such knowledge does add to our insight into the non-

convergence of the product of two such series. In his original memoir, Cesàro shewed that the product is at most simply indeterminate; if we know the indices r and s, we know also that the index of their product is at most $r+s+1$, so that if $r+s<1$, the product series is convergent. It must, however, be noted that if r or s be <-1, on account of the limitations of the multiplication theorem, it must be replaced by -1.

24. The results of § 18 and § 23 give, in combination, the result that the index of $1^{-s}-2^{-s}+3^{-s}-\ldots$ is $-s$ unattained, since if $-s+1=r>0$, $1^r-2^r+3^r-\ldots$ has index 0, and therefore

$$1^{r-1}-2^{r-1}+3^{r-1}-\ldots \equiv 1^{-s}-2^{-s}+3^{-s}-\ldots$$

has index $r-1$; if $-s+1<0$, $-s+m$ can similarly be taken (in integral), the theorem of § 18 being repeatedly applied.

By § 10, taken in conjunction with this, certain multiplication theorems due to Cajori and Pringsheim follow immediately; as, for instance, that the product of $1^{-r}-2^{-r}+3^{-r}-\ldots$ and $1^{-s}-2^{-s}+3^{-s}-\ldots$, where $r<1$, $s<1$, is a convergent series provided $r+s>1$; for the index of the product series is not greater than $-r-s+1<0$. More generally, the continued product of n series of the same type, with indices $-r_1, -r_2, \ldots, -r_n$, each r being $\leqslant 1$, is convergent provided that

$$r_1+r_2+\ldots+r_n>n-1.$$

When $r_1=r_2=\ldots=r_n$, we have another of Cajori's theorems, viz., that the n-th power of the series $1^{-s}-2^{-s}+3^{-s}-\ldots$ is convergent, provided $s>1-1/n$.

25. We now proceed to consider the summability (R, n, r) of the series $1^s-2^s+3^s-\ldots$.

(a) First we shall consider convergent series, *i.e.*, series for which s is negative; or rather, for ease in writing, we shall deal with the series $1^{-s}-2^{-s}+3^{-s}-\ldots$, where s is positive. Denoting

$$\sum_{\nu=0}^{[n]} u_\nu \left(1-\frac{\nu}{n}\right)^{-r}$$

by $\Sigma_n^{(-r)}$, we have

$$\Sigma_n^{(-r)}=\frac{n^r}{\Gamma(r)\,\Gamma(s)}\int_0^\infty\int_0^\infty x^{s-1}y^{r-1}\left[\sum_{m=1}^{[n]}(-1)^{m-1}e^{-mx}e^{-(n+1-m)y}\right]dx\,dy,$$

where a product of two repeated absolutely convergent integrals has been transformed into a double integral; this change is legitimate. The above expression is arrived at by applying the well known formula

$$\Gamma(s)\, m^{-s} \int_0^\infty e^{-mx} x^{s-1} dx,$$

to the expression of u_ν and $(n-\nu)^{-r}$.

Continuing, we have

$$\Sigma_n^{(-r)} = \frac{n^r}{\Gamma(r)\,\Gamma(s)} \int_0^\infty \int_0^\infty x^{s-1} y^{r-1} \frac{e^{-x} e^{-y}}{e^{-x}+e^{-y}} \left[e^{-ny} + (-1)^{[n]-1} e^{-nx}\right] dx\, dy$$

$$= \frac{n^r}{\Gamma(r)\,\Gamma(s)} \left\{ \int_0^\infty \left[\int_0^\infty \frac{x^{s-1} e^{-x} dx}{1+e^y e^{-x}} \right] y^{r-1} e^{-ny} dy \right.$$

$$\left. + (-1)^{[n]-1} \int_0^\infty \left[\int_0^\infty \frac{y^{r-1} e^{-y} dy}{1+e^x e^{-y}} \right] x^{s-1} e^{-nx} dx \right\},$$

changing to repeated integrals again. Now

$$\int_0^\infty \frac{x^{s-1} e^{-x}}{1+e^y e^{-x}} dx \equiv f_s(y) < \Gamma(s),$$

whatever the value of $y\ (>0)$, and

$$\int_0^\infty \frac{y^{r-1} e^{-y}}{1+e^x e^{-y}} dy \equiv f_r(x) < \Gamma(r),$$

whatever the value of $x\ (>0)$. Thus

$$\Sigma_n^{(r)} = \frac{n^r}{\Gamma(r)\,\Gamma(s)} \left[\int_0^\infty f_s(y)\, y^{r-1} e^{-ny} dy + (-1)^{[n]-1} \int_0^\infty f_r(x)\, x^{s-1} e^{-nx} dx \right].$$

Now

$$\frac{n^r}{\Gamma(r)\,\Gamma(s)} \int_0^\infty f_s(y)\, y^{r-1} e^{-ny} dy < 1,$$

whatever be the value of n. If we write it in the form

$$\frac{1}{\Gamma(r)\,\Gamma(s)} \int_0^\infty \left[\int_0^\infty \frac{x^{s-1} e^{-x}}{1+e^{y/n} e^{-x}} dx \right] y^{r-1} e^{-y} dy,$$

we see that it steadily increases with n, and hence tends to a definite limit S, say. Again,

$$\frac{n^r}{\Gamma(r)\,\Gamma(s)} \int_0^\infty \left[\int_0^\infty \frac{y^{r-1} e^{-y}}{1+e^x e^{-y}} dy \right] x^{s-1} e^{-nx} dx$$

$$= \frac{n^{r-s}}{\Gamma(r)\,\Gamma(s)} \int_0^\infty \left[\int_0^\infty \frac{y^{r-1} e^{-y}}{1+e^{x\,n} e^{-y}} dy \right] x^{s-1} e^{-x} dx.$$

As before, this repeated integral tends to a finite limit as $n \to \infty$, and consequently the whole of the last expression $\to \infty$, S, or 0 according as r is $>$, $=$, or $< s$. Thus, if $r > s$, $\Sigma_n^{(-r)}$ is infinitely oscillatory, while the series is finite $(R, n, -s)$, and summable $(R, n, -r)$ when $r < s$. The index of summability (Rn) is therefore $-s$, and the sum is

$$\lim_{n\to\infty} \frac{1}{\Gamma(r)\,\Gamma(s)} \int_0^\infty \left[\int_0^\infty \frac{x^{s-1}e^{-x}\,dx}{1+e^{y/n}e^{-x}} \right] y^{r-1}e^{-y}\,dy$$

$$= \lim_{n\to\infty} \frac{1}{\Gamma(r)\,\Gamma(s)} \int_0^\infty y^{r-1}e^{-y} f_s\left(\frac{y}{n}\right) dy.$$

Choose Y so that $$\int_Y^\infty y^{r-1}e^{-y}\,dy < \epsilon,$$

ϵ being an arbitrarily small positive number. Then

$$\frac{1}{\Gamma(r)\,\Gamma(s)} \int_Y^\infty y^{r-1}e^{-y} f_s\left(\frac{y}{n}\right) dy < \frac{\epsilon}{\Gamma(r)}.$$

Again, N_0 can be so chosen that for $n > N_0$, $0 \leqslant y \leqslant Y$,

$$\left| f_s\left(\frac{y}{n}\right) - \int_0^\infty \frac{x^{s-1}e^{-x}}{1+e^{-x}}\,dx \right| < \epsilon.$$

Thus, for $n > N_0$,

$$\left| \frac{1}{\Gamma(r)\,\Gamma(s)} \int_0^\infty y^{r-1}e^{-y} f_s\left(\frac{y}{n}\right) dy - \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}e^{-x}}{1+e^{-x}}\,dx \right| < \epsilon \left(\frac{1}{\Gamma(r)} + \frac{1}{\Gamma(s)} \right).$$

Hence the "sum" of the series is

$$\frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}e^{-x}}{1+e^{-x}}\,dx,$$

which is readily seen to equal the ordinary value of the series.

26. (*b*) We now pass on to the more difficult case when s is not negative. By using contour integrals the same method may be extended to quite general values of s (real or complex, integral or not). Instead of using the formula (in which s is supposed negative)

$$\Gamma(-s)\,m^+ = \int_0^\infty e^{-mx} x^{-s-1}\,dx,$$

we employ the expression

$$\Gamma(s)\,\frac{i}{2\pi} \int_C (-x)^{-s-1} e^{-mx}\,dx$$

for m^s. The integral is supposed to be taken along a curve commencing at $+\infty$, circulating round the origin in the counter-clockwise direction, and returning again to $+\infty$; and $(-x)^{-s}$ is to mean $e^{-s\log(-x)}$, where the real value of $\log(-x)$ is to be taken when x is negative, and the logarithm is to be rendered one-valued by the stipulation that the variable is not to cross the real axis at any point on the positive side of the origin.

All the integrals being absolutely convergent, on account of the presence of e^{-x} or e^{-y} in the integrand, the transformations used in the first part (*a*) of the investigation are valid when s has any value. It follows, by making trifling modifications in the preceding work, that quite generally the series

$$1^s-2^s+3^s-\ldots$$

is summable (R, n, r), provided that

$$R(r) > R(s);$$

R here means "real part of." And the "sum" is

$$\Gamma(s)\frac{i}{2\pi}\int_C \frac{(-x)^{-s-1}e^{-x}}{1+e^{-x}}\,dx.$$

The existence and position of the lines of summability which were introduced by Dr. H. Bohr* are rendered very evident for the case of the series considered, in the last portion of the proof.

(*c*) A slight further modification of the proof suffices to deal with the more general series

$$\Sigma n^s e^{\alpha i n}.$$

If s is negative and equal to $-s'$ (say), the expression for $\Sigma_n^{(-r)}$ which was found in part (*a*) of this section becomes

$$\frac{n^r}{\Gamma(r)\Gamma(s')}\left\{\int_0^\infty\left[\int_0^\infty \frac{x^{s'-1}e^{-(x-\alpha i)}}{1-e^y e^{-(x-\alpha i)}}\,dx\right]y^{r-1}e^{-ny}\,dy\right.$$

$$\left.-\int_0^\infty\left[\int_0^\infty \frac{y^{r-1}e^{-y}}{1-e^{x-\alpha i}e^{-y}}\,dy\right]x^{s'-1}e^{-n(x-\alpha i)}\,dx\right\}.$$

Provided that $|\sin\alpha| \geqslant \epsilon > 0$, this may be treated as in (*a*), and the

* See reference 8 in the list of memoirs appended to § 2.

series thus shewn to be summable $(R, n, -r)$, provided $r < s$, with the sum

$$\frac{e^{\alpha i}}{\Gamma(s')} \int_0^\infty \frac{x^{s'-1} e^{-x}}{1-e^{-(x-\alpha i)}} dx.$$

Moreover, for all such values of α, the convergence of $\Sigma_n^{(-r)}$ to its limit is uniform, *i.e.*, the series is uniformly summable $(R, n, -r)$ with respect to α.

In the general case, when s is not negative, for the same range of α corresponding modifications of the proof indicated in part (b) of this section may be made, and are sufficient to prove the general results relating to the series

$$\Sigma n^s \cos n\alpha,$$

$$\Sigma n^s \sin n\alpha,$$

which were stated at the end of § 23.

The expression to be considered in this last and most general case, is easily seen to be

$$-\frac{\Gamma(r)\,\Gamma(s)}{4\pi^2 n^r} \left\{ \int_{C'} \left[\int_C \frac{(-x)^{-s-1} e^{-(x-\alpha i)}}{1-e^y e^{-(x-\alpha i)}} dx \right] (-y)^{-r-1} e^{-ny} dy \right.$$

$$\left. - e^{n\alpha i} \int_{C'} \left[\int_C \frac{(-y)^{-r-1} e^{-y}}{1-e^{x-\alpha i} e^{-y}} dy \right] (-x)^{-s-1} e^{-nx} dx \right\},$$

where C is a contour of the kind described in the x-plane, and C' is a similar contour in the y-plane. The second term is oscillatory, as n increases, and the condition that it shall vanish is that

$$R(r) > R(s), \qquad 0 < \alpha < 2\pi.$$

As before, this repeated integral tends to a finite limit as $n \to \infty$, and consequently the whole of the last expression $\to \infty$, S, or 0 according as r is $>$, $=$, or $<$ s. Thus, if $r > s$, $\Sigma_n^{(-n)}$ is infinitely oscillatory, while the series is finite $(R\ \overline{-s})$, and summable $(R\ \overline{-r})$ where $r < s$. Thus the index of summability by Riesz's method is $-s$, and the sum is

$$\lim_{n \to \infty} \frac{1}{\Gamma(r)\,\Gamma(s)} \int_0^\infty \left[\int_0^\infty \frac{x^{s-1} e^{-x}}{1+e^{y^n} e^{-x}} dx \right] y^{r-1} e^{-y} dy,$$

for $0 < r < s$.

27. Some little attention has been paid to the effect on the summability and the "sum" (by Cesàro's and other methods, such as Borel's) of the

insertion of zero terms between the original terms of a series. For brevity it is convenient to speak of such insertion as "dilution" (by an obvious analogy). Such dilution may be termed *uniform* if between every pair of original terms of a series is placed a constant number of zero terms, and non-uniform when this is not the case.

Non-uniform dilution may destroy the property of Cesàro-summability of a series, as is shewn by the example (due to Mr. Hardy) quoted in § 13. Here the series $1-1+1-\ldots$ is rendered non-summable by suitable dilution. Non-uniform dilution, however, need not affect the summability or the sum. An example of this is given in Bromwich's *Infinite Series*, p. 388, Ex. 1. An example of how what we may term semi-uniform dilution may affect the sum of a series is given on p. 263 of the same treatise.

Uniform dilution, on the other hand, can affect neither the sum nor the summability of a series. For it is not difficult to prove that if Σu_n is summable (Cr), where r is integral, then the uniformly diluted series Σv_n, obtained by inserting $(m-1)$ zero terms between each original pair, is also summable (Cr). A proof can easily be based on equation (2) of § 124 of Dr. Bromwich's treatise. Assuming this theorem on the summability, the fact that

$$\Sigma v_n x^n = \Sigma u_n x^{mn},$$

and

$$\Sigma u_n x^n,$$

have the same limiting value as $x \to 1$, shews that the two sums are the same.

Though I have not troubled to write out a formal proof, the above theorem on summability of a diluted series can probably be extended to the case when r is not integral.

For further remarks on the summability of diluted series we may refer to § 29 of the paper (15) cited in § 2.

28. THEOREM.—*An absolutely convergent series* Σu_n, *such that* $nu_n \to 0$ *as* $n \to \infty$, *has an index of summability* $\not> -1$.

Evidently all absolutely convergent series of terms arranged in descending order of magnitude come under this theorem. The proof is not difficult, but as the theorem holds not only for Cesàro's method of summation, but also for Dr. Riesz's much more general method, a general proof will be given.

To make the one proof suffice for Cesàro's and Riesz's methods, we

shall consider the limit of

$$\Sigma_n^{(r,\lambda)} = \sum_{m=1}^{n} u_m p_{n-m} \left(1 - \frac{\lambda_m}{\lambda_n}\right)^r,$$

where p_{n-m} is unity if Riesz's mode of summation be considered, while it tends to unity as $n \to \infty$ if Cesàro's mode be used (in which case $\lambda_m = m$).

Consider the first θn terms, where $0 < \theta < 1$; by Tannery's theorem their sum tends to the sum of the series, as $n \to \infty$. Now the p's are always finite, and moreover $nu_n \to 0$ as $n \to \infty$, if μ is the function inverse to λ, *i.e.*, if

$$\lambda_m = M, \quad m = \mu(M).$$

Hence the sum of the remaining terms of $\Sigma_n^{(r,\lambda)}$ is less than

$$K\epsilon_n \int_\theta^1 f(x)(1-x)^r \frac{dx}{x};$$

here K denotes a constant independent of n, $\epsilon_n \to 0$ as $n \to \infty$, and

$$f(x) = \frac{M}{\mu(M)} = \frac{\lambda_m}{m}, \qquad x = \frac{M}{N}.$$

Thus, if $\overline{\lim_{n\to\infty}}\ (\lambda_m/m)$ be finite, the sum of the remaining terms of $\Sigma_n^{(r,\lambda)}$ tends to zero as $n \to \infty$, provided that $r > -1$; the result follows.

It may be noticed here that the same theorem holds for summable integrals.

If the condition $nu_n \to 0$ as $n \to \infty$ is not satisfied, by § 11 the index must be $\geqslant -1$. It is evident that we can so dilute an absolutely convergent series as to make the transformed series* $\Sigma u'_n$ such that $n^k u'_n$ does not tend to 0 as $n \to \infty$, for any positive value of k whatever. The transformed series, being convergent, has by the theorem of § 11 the index 0 attained.

The general multiplication theorem of § 10 would only give

$$0+0+1 = 1$$

as the upper limit of the possible values of the index of the product series formed from two such transformed absolutely convergent series. By Cauchy's theorem it is thus evident that in this case the upper limit so given is greater than the actual value of the index by at least unity.

In conclusion, we may remark that series may be found with as large

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* Choose any sequence of positive numbers $k_0, k_1, k_2, \ldots$ tending to 0, and any sequence of positive integers $b_0, b_1, b_2, \ldots$, such that $b_n^{k_n} u_n \to \infty$ as $n \to \infty$. Then dilute the series so that $u'_{b_n} = u_n$, all other values of u'_n being 0.

a negative index as we please, or even with $-\infty$ for index. It is not difficult to prove that if $|n^t u_n| < K$ for all values of n, where $t > 1$, the index of Σu_n is $\not> -t$; while the geometrical series $1-x+x^2-\ldots$, where $|x| < 1$, has $-\infty$ for its index, *i.e.*, is summable $(C\overline{-k})$ however large k may be.

PART IV.

Summable Integrals.

29. The extension of Cesàro's methods to the summation of infinite integrals was first considered by Mr. Hardy;* only a first "mean" of an integral was there taken, however. Very shortly afterwards Dr. C. N. Moore published a paper† on the same subject—also considering only summation $(C1)$—in which he proved many properties of integrals so summable; in particular he dealt with the introduction of convergence factors into integrals summable $(C1)$.

The general extension is obvious. Thus an integral is said to be summable (Cr) if the limit as $x \to \infty$ of

$$\frac{r!}{x^r}\int_a^x\int_a^{a_1}\int_a^{a_2}\ldots\int_a^{a} f(\xi)\,d\xi\,da_r\,da_{r-1}\ldots da,$$

exists and is finite; r is necessarily a positive integer here. The above repeated integral can immediately be transformed into

$$\int_a^x f(\beta)\left(1-\frac{\beta}{x}\right)^r d\beta,$$

which is evidently analogous to Dr. Riesz's method of summation of series, for the case $\lambda_n = n$. (Mr. Hardy has considered the integral analogue of Dr. Riesz's most general method of summation.‡) In the latter form r need not be integral, and consequently this form is adopted as the basis of the general theory of summable integrals.

30. I shall first state the theorem for integrals analogous to Cesàro's

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* *Quarterly Journal of Mathematics*, Vol. 35, p. 54.

† *Trans. Amer. Math. Soc.*, Vol. 8, 1907.

‡ I think it extremely probable that Dr. Riesz has himself considered the same problem, though I have not seen any reference to it in his published works.

theorem for sequences, which was used in § 8.

If $\frac{f(x)}{x^r} \to F$, $\frac{g(x)}{x^s} \to G$, *as* $x \to \infty$, *then provided that* $r > -1$, $s > -1$,

$$\lim_{x \to \infty} \frac{1}{x^{r+s+1}} \int_a^x f(t)\, g(x-t)\, dt,$$

is equal to

$$\frac{\Gamma(r+1)\,\Gamma(s+1)}{\Gamma(r+s+2)}\, FG.$$

Here $f(x)$ and $g(x)$ need not be continuous. As far as I am aware, the above theorem is new. Its proof presents no difficulty, and will not be set out here.

The first theorem required in this subject is the one analogous to that of § 8, and which shews, of course, that in the case of convergent integrals the condition of consistency is fulfilled.

If the integral $\int_a^\infty \phi(x)\, dx$ *is summable* (Cr), *with sum* S, r *being any real number* > -1, *integral or not, then it is summable* (Cr') *with the same sum, provided that* $r' > r$.

This theorem has been established independently by Mr. Hardy for $r > 0$, by a somewhat similar method of proof.

Write $r' = r+s+1$, so that $s > -1$. In the analogue of Cesàro's theorem, substitute

$$f(x) = \int_a^x \phi(\beta)\,(x-\beta)^r\, d\beta,$$

$$g(x) = x^s.$$

Then $f(x)/x^r \to S$, $g(x)/x^s \to 1$, as $x \to \infty$. Hence

$$\frac{1}{x^{r+s+1}} \int_a^x (x-\alpha)^s \int_a^\alpha \phi(\beta)\,(\alpha-\beta)^r\, d\beta\, d\alpha,$$

tends as $x \to \infty$ to

$$\frac{\Gamma(r+1)\,\Gamma(s+1)}{\Gamma(r+s+2)}\, S.$$

If $\phi(\beta)$ is continuous we may evidently invert the order of integration, and obtain

$$\frac{1}{x^{r+s+1}} \int_a^x \phi(\beta) \int_\beta^x (\alpha-\beta)^r\,(x-\alpha)^s\, d\alpha\, d\beta$$

$$= \frac{\Gamma(r+1)\,\Gamma(s+1)}{\Gamma(r+s+2)}\, \frac{1}{x^{r+s+1}} \int_a^x \phi(\beta)\,(x-\beta)^{r+s+1}\, d\beta.$$

Hence $$S = \lim_{x\to\infty} \frac{1}{x^{r+s+1}} \int_a^x \phi(\beta)(x-\beta)^{r+s+1}\,d\beta.$$

Since $r+s+1 = r'$, this proves the theorem.

31. The well known theorems of Cauchy, Abel, Mertens, and Cesàro, on the multiplication of series, have interesting analogues for integrals. Mr. Hardy has shewn* that if $a(x)$, $b(x)$ be continuous functions of x, such that

$$A(x) = \int_0^x a(x)\,dx, \qquad B(x) = \int_0^x b(x)\,dx,$$

have definite limits as $x \to \infty$, then if

$$c(x) = \int_0^x a(u)\,b(x-u)\,du,$$

the integral $$C(x) = \int_0^x c(x)\,dx,$$

is at any rate simply summable, if not convergent.

This theorem is a particular case of the following:—*if* $\int_0^\infty a(x)dx$ *is summable* (Cr), *with sum* A, *and* $\int_0^\infty b(x)dx$ *is summable* (Cs), *with sum* B, *then if* $r > -1$, $s > -1$, *the integral* $\int_0^\infty c(x)dx$ *is summable* $(C\,\overline{r+s+1})$, *with sum* AB.

In the analogue of Cesàro's theorem, write

$$f(x) = \int_0^x a(u)(x-u)^r\,du,$$

$$g(x) = \int_0^x b(u)(x-u)^s\,du.$$

Then $$f(x)/x^r \to A, \qquad g(x)/x^s \to B.$$

Therefore $$\frac{1}{x^{r+s+1}} \int_0^x f(z)\,g(x-z)\,dz$$

has the limit $$\frac{\Gamma(r+1)\,\Gamma(s+1)}{\Gamma(r+s+1)}\,AB,$$

* *Proc. London Math. Soc.*, Ser. 2, Vol. 8, p. 301.

as $x \to \infty$. Now

$$\int_0^x f(z)\, g(x-z)\, dz = \int_0^x \int_0^z a(y)(z-y)^r\, dy \int_0^{x-z} b(u)(x-z-u)^s\, du\, dz$$

$$= \int_0^x b(u)\, du \int_0^{x-u} a(y)\, dy \int_y^{x-u} (z-y)^r\, (x-z-u)^s\, dz,$$

changing the order of integration, which is evidently legitimate under the stated conditions.

Put $$\theta = \frac{z-y}{x-u-y}.$$

We get

$$\int_0^x f(z)\, g(x-z)\, dz = \frac{\Gamma(r+1)\,\Gamma(s+1)}{\Gamma(r+s+2)} \int_0^x b(u)\, du \int_0^{x-u} a(y)(x-u-y)^{r+s+1}\, dy,$$

if $r > -1$, $s > -1$. Again invert the order of integration. Using our former result, we have

$$\lim_{x\to\infty} \frac{1}{x^{r+s+1}} \int_0^x (x-y)^{r+s+1}\, dy \int_0^y b(u)\, a(y-u)\, du$$

$$= \lim_{x\to\infty} \frac{1}{x^{r+s+1}} \int_0^x (x-y)^{r+s+1}\, c(y)\, dy$$

$$= AB,$$

which proves the theorem.

A CLASS OF ORTHOGONAL SURFACES WITH THE PROPERTY THAT FROM ANY ONE OF THEM AN INFINITE NUMBER OF OTHERS CAN BE DEDUCED BY DIFFERENTIATION ALONE

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I wish here to discuss the fundamental property of a particular class of orthogonal surfaces only; but, in order to be the more easily intelligible and to make clear the relation of this particular class of surfaces to the general orthogonal surfaces, it will be advisable to sketch, as rapidly as possible, the main outlines (at least, so far as they bear on our subject) of the general theory as developed by Darboux, in his *Theory of Surfaces*, and by Bianchi, in his *Differential Geometry*.

The substance, if not the form, of the first few pages of this paper must then be considered well known.

If two different systems of triply orthogonal surfaces can have a point correspondence established between them, of such a nature that at any point of the one system the normals are parallel to the normals at the corresponding point of the other system, then any surface in one system will have the same spherical representation as the corresponding surface in the other system; and we shall say that the systems are "parallel." Now, in general, given any orthogonal system, the determination of the parallel systems depends on the solution of certain simultaneous linear partial differential equations of the second order fundamental in the theory of surfaces; but in the particular class of surfaces here considered these equations will have the property that from any set of solutions we can obtain an infinity of others by differentiation alone.

A like property was found by Bäcklund to hold for the equation which defines the surfaces applicable on a given pseudosphere, and a rather laborious paper by the author in the *Proceedings* would seem to show how very limited in number are the equations characterized by such a property. If the class of orthogonal surfaces here considered is transformed by the beautiful contact transformation which Lie discovered, and which has the property of transforming spheres into straight lines, we

shall obtain new systems of surfaces any one of which will have an infinity of known infinitesimal deformations.

1. Let λ, μ, ν be three unit vectors mutually at right angles and depending on three parameters u, v and w.

Let the suffix 1 attached to a vector or scalar denote the vector or scalar differentiated with respect to u, and let the suffixes 2 and 3 have similar meanings with respect to v and to w.

Just as in the case where these vectors depend on two parameters only, we now have

$$\left.\begin{aligned} \lambda_1 &= \mu r' - \nu q', & \mu_1 &= \nu p' - \lambda r', & \nu_1 &= \lambda q' - \mu p' \\ \lambda_2 &= \mu r'' - \nu q'', & \mu_2 &= \nu p'' - \lambda r'', & \nu_2 &= \lambda q'' - \mu p'' \\ \lambda_3 &= \mu r''' - \nu q''', & \mu_3 &= \nu p''' - \lambda r''', & \nu_3 &= \lambda q''' - \mu p''' \end{aligned}\right\}, \qquad (1)$$

where the scalar functions thus introduced have the geometrical meaning that

$$p'du+p''dv+p'''dw, \quad q'du+q''dv+q'''dw, \quad r'du+r''dv+r'''dw$$

are the angular displacements of the set of axes formed by the vectors λ, μ, and ν.

Let now z be a vector drawn from the origin and depending on the same three parameters; and suppose that the surfaces which z traces out, when u, v and w respectively are constant, form a triply orthogonal system with their normals respectively parallel to λ, μ, and ν.

We have

$$S\lambda z_2 = S\lambda z_3 = 0, \quad S\mu z_3 = S\mu z_1 = 0, \quad S\nu z_1 = S\nu z_2 = 0;$$

and therefore $$z_1 = \xi\lambda, \quad z_2 = \eta\mu, \quad z_3 = \zeta\nu,$$

where ξ, η, and ζ are scalar functions. From the equivalence of z_{12} and z_{21} we see that

$$\xi_2 = -\eta r', \quad \xi r'' = \eta_1, \quad \xi q'' = -\eta p'.$$

Similarly we have

$$\xi_3 = \zeta q', \qquad \zeta_1 = -\xi q''', \quad \xi r''' = -\zeta p',$$
$$\eta_3 = -\zeta p'', \quad \zeta_2 = \eta p''', \qquad \zeta q'' = -\eta r'''.$$

It follows that $$p' = 0, \quad q'' = 0, \quad r''' = 0;$$

and therefore we easily see that

$$S\lambda z_{23} = 0, \quad S\mu z_{31} = 0, \quad S\nu z_{12} = 0.$$

Now, from the fact that

$$S\lambda z_{23} = 0, \quad S\lambda z_2 = 0, \quad S\lambda z_3 = 0,$$

we see that on the surface $u =$ constant, the curves $v =$ constant and $w =$ constant are conjugate; and therefore, as they are orthogonal curves also, they are lines of curvature, so that we have the fundamental theorem that the lines of intersection of the surfaces are their lines of curvature.

The condition that the equations satisfied by the vectors λ, μ, ν may be consistent are easily found: they are

$$p_3''-p_2''' = q''r'''-q'''r'', \quad q_3''-q_2''' = r''p'''-r'''p'',$$
$$r_3''-r_2''' = p''q'''-p'''q'',$$

with six other similar ones.

Introducing a notation like Darboux's, we now write

$$q' = (31), \qquad r'' = (12), \qquad p''' = (23),$$
$$r' = -(21), \quad p'' = -(32), \quad q''' = -(13).$$

Remembering that $p' = q'' = r''' = 0$, the equations of compatibility may now be written

$$\left.\begin{aligned}(23)_2+(32)_3+(12)(13) = 0, \quad (31)_3+(13)_1+(23)(21) = 0\\ (12)_1+(21)_2+(31)(32) = 0\end{aligned}\right\}, \qquad \text{(A)}$$

$$\left.\begin{aligned}(23)_1 = (21)(13), \quad (31)_2 = (32)(21), \quad (12)_3 = (13)(32)\\ (32)_1 = (31)(12), \quad (13)_2 = (12)(23), \quad (21)_3 = (23)(31)\end{aligned}\right\}, \qquad \text{(B)}$$

where the suffixes have the usual meaning of differentiation.

Conversely it may be shown that any six functions satisfying these nine equations—rotation functions we shall call them—will define the motion of three rectangular axes parallel to the normals of a triply orthogonal system; the unit vectors λ, μ, ν along these axes may, in fact, be determined by the solution of an equation of Riccati's form as is shown in § 2 of the author's paper on "Cyclic Congruences."

When, however, we have determined these vectors, we have not determined the orthogonal system, as we must first find ξ, η, and ζ.

Surfaces with the same "rotation" functions will have their normals parallel at corresponding points, or by a mere movement can be placed so as to have this property.

When ξ, η, and ζ are determined, we can find z by quadratures from

the equations

$$z_1 = \xi\lambda, \quad z_2 = \eta\mu, \quad z_3 = \zeta\nu.$$

The equations which, when the rotation functions are known, determine ξ, η, and ζ are

$$\left.\begin{aligned} \xi_2 &= \eta\,(21), \quad \eta_3 = \zeta\,(32), \quad \xi_1 = \xi\,(13) \\ \xi_3 &= \zeta\,(31), \quad \eta_1 = \xi\,(12), \quad \xi_2 = \eta\,(23) \end{aligned}\right\}, \tag{2}*$$

From these equations we see that

$$\left.\begin{aligned} \xi_{12} &= \xi_1 \frac{\partial}{\partial v}\log(13) + \xi_2 \frac{\partial}{\partial u}\log(23) \\ \xi_{13} &= \xi_1 \frac{\partial}{\partial w}\log(13) + \xi\,(31)(13) \\ \xi_{23} &= \xi_2 \frac{\partial}{\partial w}\log(23) + \xi\,(32)(23) \end{aligned}\right\}, \tag{2}'$$

these three equations together with

$$\xi = \frac{\xi_1}{(13)}, \qquad \eta = \frac{\xi_2}{(23)},$$

being equivalent to the first six.

The problem of finding orthogonal systems may then be taken to consist of three parts: the first part is the finding of rotation functions, the second is the solution of the above equations, the third is the determination of z from

$$z_1 = \xi\lambda, \qquad z_2 = \eta\mu, \qquad z_3 = \zeta\nu.$$

Instead of finding the orthogonal surfaces in this way, however, we might proceed as follows.

Let $$z = \xi'\lambda + \eta'\mu + \zeta'\nu,$$

where ξ', η', and ζ' are scalars: remembering that

$$\left.\begin{aligned} &\lambda_1 + \mu(21) + \nu(31) = 0, \quad \mu_2 + \nu(32) + \lambda(12) = 0, \quad \nu_3 + \lambda(13) + \mu(23) = 0 \\ &\lambda_2 = \mu(12), \qquad \mu_1 = \lambda(21), \qquad \nu_1 = \lambda(31) \\ &\lambda_3 = \nu(13), \qquad \mu_3 = \nu(23), \qquad \nu_2 = \mu(32) \end{aligned}\right\}, \tag{2}''$$

* The part of Darboux's *Surfaces* to which I would refer as regards the above discussion is Chap. xii, Vol. iv. His H_1, H_2, H_3 are my ξ, η, and ζ, his $\beta_{\iota\kappa}$ is my (ι, κ). He does not use any vector notation, nor does he interpret his functions $\beta_{\iota\kappa}$ as rotations.

and that $z_1 = \xi\lambda, \quad z_2 = \eta\mu, \quad z_3 = \zeta\nu,$

we see that we must have

$$\left.\begin{aligned} \xi'_2 &= \eta'(12), \quad \eta'_3 = \zeta'(23), \quad \zeta'_1 = \xi'(31) \\ \xi'_3 &= \zeta'(13), \quad \eta'_1 = \xi'(21), \quad \zeta'_2 = \eta'(32) \end{aligned}\right\}. \tag{3}$$

We also have

$$\xi = \xi'_1 + \eta'(21) + \zeta'(31), \quad \eta = \eta'_2 + \zeta'(32) + \xi'(12), \quad \zeta = \zeta'_3 + \xi'(13) + \eta'(23). \tag{3}'$$

Conversely, if ξ', η', and ζ' are any three scalar functions which satisfy the equations (3), we see that

$$z = \xi'\lambda + \eta'\mu + \zeta'\nu,$$

will give a triply orthogonal system of surfaces with λ, μ, and ν parallel to the normals at each point.

It will be noticed that the compatibility of the simultaneous equations (3) depend only on the equation system (B), not on the equations (A). From (3)′ we see also that from any set of solutions of (3) we can obtain a set of solutions of (2) by differentiation alone. The particular class of surfaces which forms the subject of this paper will have the further property that from any set of solutions of (3) an infinity of others can be derived by differentiation only.

From (3) we deduce

$$\zeta'_{12} = \zeta'_1 \frac{\partial}{\partial v} \log(31) + \zeta'_2 \frac{\partial}{\partial u} \log(32),$$

an equation fundamental in the theory of the surface $w =$ constant, and one with the property—if the surface belongs to this particular class—that from any solution of it an infinity of other solutions can be derived by differentiation only.

The equation system (3) is, in fact, equivalent with the system

$$\left.\begin{aligned} \zeta'_{12} &= \zeta'_1 \frac{\partial}{\partial v} \log(31) + \zeta'_2 \frac{\partial}{\partial u} \log(32) \\ \zeta'_{13} &= \zeta'_1 \frac{\partial}{\partial w} \log(31) + \zeta'(31)(13) \\ \zeta'_{23} &= \zeta'_2 \frac{\partial}{\partial w} \log(32) + \zeta'(32)(23) \end{aligned}\right\}, \tag{4}$$

$$\xi' = \frac{\zeta'_1}{(31)}, \qquad \eta' = \frac{\zeta'_2}{(32)}.$$

2. The above may be taken as a brief account of the general problem in connection with the determination of orthogonal surfaces, and, before passing on to the special system I wish to consider, I would like to make a few general remarks on it.

First, it should be noticed that to any two sets of solutions of the equations (2) there will correspond two orthogonal systems in which the normals will be parallel at corresponding points.

Take any two corresponding surfaces, the normals being parallel along lines of curvature at corresponding points, the surfaces are said to have the same spherical representation; and therefore, as Darboux has shown in Chap. viii of Vol. IV of his treatise, if these surfaces are transformed by the contact transformation which turns spheres into straight lines, we shall be given infinitesimal deformations for each of the transformed surfaces. But if we are given *any* orthogonal system, even if we are not given a "parallel" orthogonal system, we can find* three systems of surfaces having the same spherical representation as the given orthogonal system; and therefore we can find infinitesimal deformations for the surfaces into which the orthogonal system is transformed by this contact transformation.

To see this, consider the equations (B) which the rotation functions satisfy. Since

$$\frac{\partial}{\partial u}(23)(32) = \frac{\partial}{\partial v}(31)(13) = \frac{\partial}{\partial w}(12)(21),$$

we see that we can find a function V of the parameters u, v, and w, such that

$$(23)(32) = V_{23}, \quad (31)(13) = V_{13}, \quad (12)(21) = V_{12},$$

and Darboux remarks, § 1041, that we can express all the rotation functions in terms of V, and find the differential equations which V satisfies. This is no doubt true, and I have tried to do this, but the labour involved seemed prohibitive of their actual calculation in the general case.

Consider now the surfaces given by

$$z = \lambda V_1 - \lambda_1.$$

Where u is given any constant value, I say that this surface has the same spherical representation as the corresponding surface $u =$ constant in the given orthogonal system.

* This also follows from the footnote Bianchi-Lukat, *Differential Geometrie*, p. 345, which connects the equation giving the spherical representations of a surface with the equation which ζ satisfies when ζdw is the normal distance between two adjacent surfaces of an orthogonal system. It was this remark which led me to the direct determination of the required system.

To prove this, notice that

$$\begin{aligned} z_2 &= \lambda V_{12} + \lambda_2 V_1 - \frac{\partial}{\partial u}\lambda_2 \\ &= \lambda V_{12} + \mu(12) V_1 - \frac{\partial}{\partial u}\mu(12) \text{ by (2)}'' \\ &= \lambda V_{12} + \mu(12) V_1 - \lambda(21)(12) - \mu \frac{\partial}{\partial u}(12) \\ &= \mu\left((12) V_1 - \frac{\partial}{\partial u}(12)\right) \\ &= \lambda_2 \frac{\partial}{\partial u}\left(V - \log(12)\right). \end{aligned}$$

Similarly we see that

$$z_3 = \lambda_3 \frac{\partial}{\partial u}\left(V - \log(13)\right);$$

and therefore the lines $v =$ constant and $w =$ constant are lines of curvature on this surface, and λ is parallel to its normal.

The other two systems of surfaces would similarly be found to be

$$z = \mu V_2 - \mu_2 \quad \text{and} \quad z = \nu V_3 - \nu_3.$$

It is not, of course, asserted that these surfaces are orthogonal surfaces. It is rather remarkable that Lie's contact transformation should from any given orthogonal system enable us to obtain a system of surfaces each with a known infinitesimal deformation, and that this fact which follows so easily from what Darboux has proved should not, so far as I know, have been noticed.

Thus let us take Dupin's "Cyclide" (Salmon, *Geometry of Three Dimensions*, p. 535), and form (*ibid.*, p. 533) the system of cyclides confocal with it. These form a triply orthogonal system. Now we know (see author's *Continuous Groups*, § 206) that the contact transformation which converts spheres into straight lines will transform the cyclide into a quadric. It follows, then, that for any quadric we know an infinitesimal deformation.

3. So far we have been considering the general orthogonal system and assuming that the rotation functions are the most general solutions of the equations (A) and (B). A special orthogonal system will be obtained by assuming the rotation functions to satisfy some other equation or equations than merely (A) and (B).

Obviously it will not do to assume these equations arbitrarily; they must be equations which are compatible with the equations (A) and (B). I shall first assume that the equations

$$\left.\begin{aligned}(23)_3+(32)_2+(21)(31)=0,\quad (31)_1+(13)_3+(32)(12)=0\\ (12)_2+(21)_1+(13)(23)=0\end{aligned}\right\}, \qquad (C)$$

are compatible with (A) and (B), and deduce the properties of the corresponding orthogonal system, and afterwards I shall justify this assumption and find the equation system on which (A), (B), and (C) depend.

Assuming then that the rotation functions satisfy the conditions (A), (B), and (C), let

$$z=\xi'\lambda+\eta'\mu+\zeta'\nu$$

be any triply orthogonal system of the family, we shall show how a second system can be deduced, and on on, *ad infinitum*.

The functions ξ', η', and ζ' satisfy the equations (3); let

$$\xi=\xi'_1+\eta'(21)+\zeta'(31),\quad \eta=\eta'_2+\zeta'(32)+\xi'(12),\quad \zeta=\zeta'_3+\xi'(13)+\eta'(23),$$

be functions derived from ξ', η', and ζ'; and let

$$\xi''=\xi_1+\eta(12)+\zeta(13),\quad \eta''=\eta_2+\zeta(23)+\xi(21),\quad \zeta''=\zeta_3+\xi(31)+\eta(32).$$

We now see, by aid of (B), (C), and (2), that the functions ξ'', η'', and ζ'' satisfy the equations (3); and therefore

$$z=\xi''\lambda+\eta''\mu+\zeta''\nu$$

is another orthogonal system with the same rotation functions.

Proceeding similarly with the new system, we should obtain a third orthogonal system, and so on.

In order to be sure that we really are obtaining new systems by this method it is only necessary to prove that ζ'', say, is not equal to ζ'.

Now $$\zeta''=\xi_3+\xi(31)+\eta(32)=\xi_3+\xi_1\frac{(31)}{(13)}+\xi_2\frac{(32)}{(23)},$$

by (2), and $$\xi=\xi'_3+\xi'(13)+\eta'(23)=\xi'_3+\xi'_1\frac{(13)}{(31)}+\xi_2\frac{(23)}{(32)},$$

by (4); and therefore

$$\zeta''=\left(\frac{(31)}{(13)}\frac{\partial}{\partial u}+\frac{(32)}{(23)}\frac{\partial}{\partial v}+\frac{\partial}{\partial w}\right)\left(\frac{(13)}{(31)}\frac{\partial}{\partial u}+\frac{(23)}{(32)}\frac{\partial}{\partial v}+\frac{\partial}{\partial w}\right)\zeta'.$$

As (4) are the only equations of the second order which ζ' must satisfy, we see that in general ζ'' cannot be equal to ζ'.

If, then, any of these special systems be transformed by the contact transformation we have been considering, we shall have a system of surfaces for any one of which an infinite number of infinitesimal deformations will be known.

Again, from any solution of the equation system (3), we can deduce a second, and so on, *ad infinitum*.

We can also deduce an infinity of solutions of (2) from any one of (2) or of (3), or of the system (2)′ equivalent with (2).

It will be noticed that we have really got two sets of rotation functions, viz.,

$$q' = (31), \quad r'' = (12), \quad p''' = (23),$$
$$r' = -(21), \quad p'' = -(32), \quad q''' = -(13),$$

and

$$q' = (13), \quad r'' = (21), \quad p''' = (32),$$
$$r' = -(12), \quad p'' = -(23), \quad q''' = -(31),$$

and, given any system corresponding to one of these, we can deduce an infinity of surfaces corresponding to either.

4. We now proceed to consider the equations which will define the required system of rotation functions.

Let $$2u' = v+w, \quad 2v' = w+u, \quad 2w' = u+v,$$

and if f is any function of the parameters u', v', w', let us denote

$$\frac{\partial f}{\partial u'}, \quad \frac{\partial f}{\partial v'}, \quad \frac{\partial f}{\partial w'}, \quad -\frac{\partial f}{\partial u'} - \frac{\partial f}{\partial v'} - \frac{\partial f}{\partial w'},$$

respectively by $$f_1, \quad f_2, \quad f_3, \quad f_4,$$

with similar interpretations of the symbols f_{12}, f_{14}, f_{123}, and so on.

First we prove that the equations

$$f_{234} + 4f_{23}^{\frac{1}{2}} f_{34}^{\frac{1}{2}} f_{42}^{\frac{1}{2}} = 0,$$
$$f_{341} + 4f_{34}^{\frac{1}{2}} f_{41}^{\frac{1}{2}} f_{13}^{\frac{1}{2}} = 0,$$
$$f_{412} + 4f_{41}^{\frac{1}{2}} f_{12}^{\frac{1}{2}} f_{24}^{\frac{1}{2}} = 0,$$
$$f_{123} + 4f_{12}^{\frac{1}{2}} f_{23}^{\frac{1}{2}} f_{31}^{\frac{1}{2}} = 0,$$

form a complete system—that is, by differentiation and elimination no other equation of the third or lower order could be derived from them; and therefore, of course, they are compatible.

Differentiating the first with respect to u', the second with respect to v', the third with respect to w', and on the last performing the operation

$$-\frac{\partial}{\partial u'}-\frac{\partial}{\partial v'}-\frac{\partial}{\partial w'},$$

we get in each case the equation

$$f_{1234}=8\,(f_{12}^{\frac{1}{2}}f_{34}^{\frac{1}{2}}f_{13}^{\frac{1}{2}}f_{24}^{\frac{1}{2}}+f_{13}^{\frac{1}{2}}f_{24}^{\frac{1}{2}}f_{23}^{\frac{1}{2}}f_{14}^{\frac{1}{2}}+f_{23}^{\frac{1}{2}}f_{14}^{\frac{1}{2}}f_{12}^{\frac{1}{2}}f_{34}^{\frac{1}{2}}),$$

and thus we see that the system is complete.

Take now $$x=f_{23}^{\frac{1}{2}},\quad y=f_{31}^{\frac{1}{2}},\quad z=f_{12}^{\frac{1}{2}},$$
$$x'=f_{14}^{\frac{1}{2}},\quad y'=f_{24}^{\frac{1}{2}},\quad z'=f_{34}^{\frac{1}{2}},$$

and we see that $$x_1=-2yz,\quad x_4=-2y'z',$$
$$y_2=-2zx,\quad y_4=-2z'x',$$
$$z_3=-2xy,\quad z_4=-2x'y',$$
$$x_2'=-2y'z,\quad y_1'=-2x'z,\quad z_1'=-2x'y,$$
$$x_3'=-2z'y,\quad y_3'=-2z'x,\quad z_2'=-2y'x.$$

Returning to the original variables u, v, w,

$$\frac{\partial}{\partial u'}=\frac{\partial}{\partial v}+\frac{\partial}{\partial w}-\frac{\partial}{\partial u},$$
$$\frac{\partial}{\partial v'}=\frac{\partial}{\partial w}+\frac{\partial}{\partial u}-\frac{\partial}{\partial v},$$
$$\frac{\partial}{\partial w'}=\frac{\partial}{\partial u}+\frac{\partial}{\partial v}-\frac{\partial}{\partial w}.$$

These equations become—the suffixes having their usual meaning—

$$x_2+x_3-x_1=-2yz,\qquad x_1+x_2+x_3=2y'z',$$
$$y_3+y_1-y_2=-2zx,\qquad y_1+y_2+y_3=2z'x',$$
$$z_1+z_2-z_3=-2xy\qquad z_1+z_2+z_3=2x'y',$$
$$x_3'+x_1'-x_2'=-2y'z,\qquad y_2'+y_3'-y_1'=-2x'z,$$
$$x_1'+x_2'-x_3'=-2z'y,\qquad y_1'+y_2'-y_3'=-2z'x,$$
$$z_2'+z_3'-z_1'=-2x'y,\qquad z_3'+z_1'-z_2'=-2y'x;$$

or
$$x_2+x_3 = y'z'-yz, \qquad x_1 = \quad y'z'+yz,$$
$$x_2'-x_3' = y'z-z'y, \qquad x_1' = -z'y-y'z,$$
$$y_3+y_1 = z'x'-zx, \qquad y_2 = \quad z'x'+zx,$$
$$y_3'-y_1' = z'x-x'z, \qquad y_2' = -x'z-z'x,$$
$$z_1+z_2 = x'y'-xy, \qquad z_3 = \quad x'y'+xy,$$
$$z_1'-z_2' = x'y-y'x, \qquad z_3' = -y'x-x'y.$$

If we now take
$$(23) = x+x', \qquad (31) = y+y', \qquad (12) = z+z',$$
$$(32) = x-x', \qquad (13) = y-y', \qquad (21) = z-z',$$

we see that
$$(23)_2+(32)_3+(13)(12) = 0, \qquad (23)_3+(32)_2+(31)(21) = 0,$$
$$(31)_3+(13)_1+(23)(21) = 0, \qquad (31)_1+(13)_3+(32)(12) = 0,$$
$$(12)_1+(21)_2+(31)(32) = 0, \qquad (12)_2+(21)_1+(13)(23) = 0,$$
$$(23)_1 = (21)(13), \qquad (31)_2 = (32)(21), \qquad (12)_3 = (13)(32),$$
$$(32)_1 = (31)(12), \qquad (13)_2 = (12)(23), \qquad (21)_3 = (23)(31).$$

That is, (23), (32), (31), (13), (12), (21) are rotation functions satisfying the conditions (A), (B), and (C).

ON THE CONDITIONS THAT A TRIGONOMETRICAL SERIES SHOULD HAVE THE FOURIER FORM

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1. While the study of power series in general is an essential preliminary to the discussion of the special power series which occurs in Taylor's theorem, our knowledge of trigonometrical series has been chiefly obtained indirectly from considerations connected with the theory of the expansion of a function in a series of Fourier. A few theorems concerning trigonometrical series, obtained independently of such considerations, are, however, now in existence. Thus Harnack, Georg Cantor, Riemann, and others have shewn that, under certain circumstances connected with the convergence of the series, the coefficients must necessarily tend towards zero as unique limit. Their results are included in the following general statement:—If the points at which a trigonometrical series does not converge form a set of the first category, the coefficients converge to zero.*

From this it follows† that no two distinct trigonometrical series can exist whose sums agree except at a countable set of points. More generally, Bernstein, employing a different method, has shewn that the sums cannot agree everywhere except at points forming a set having no perfect component.‡ On the other hand, Fatou§ has proved, by considerations connected with Fourier series and Poisson's integral, that, if the coefficients have, when multiplied each by the integer which indicates its place in the series, the unique limit zero, the series converges except at most at a set of content zero.

Perhaps, however, the most striking results in this department are those associated with the name of du Bois Reymond,‖ who proved that, if a trigonometrical series has a sum, then, provided its coefficients con-

* W. H. Young, "A Note on Trigonometrical Series" (1908), *Messenger of Mathematics*, Vol. XXXVIII, pp. 44–48.

† *Ibid.*

‡ F. Bernstein, *Zur Theorie der trigonometrischen Reihen*, 1908, Leipzig, Ber., pp. 325–33.

§ P. Fatou, "Séries Trigonométriques et séries de Taylor" (1906), *Acta Math.*, Vol. XXX, pp. 335–400. [*Note added May 1st*, 1911.—This result has been further extended by Jerosch and by Weyl, *Math. Ann.*, Vols. 66, 67.]

‖ P. du Bois Reymond, *Abhandlungen der baierischen Academie*, Bd. X (1875).

verge to zero, the series is necessarily expressible in the Fourier form whenever the coefficients have a meaning in the Riemann sense.

This last result has been extended so as to include certain cases in which the integrals in question exist as Lebesgue or Harnack-Lebesgue integrals; but, as far as I know, no further advance has been made.*

The object of the present paper is to prove that *every trigonometrical series whose upper and lower functions are bounded is necessarily a Fourier series;* more generally, that this remains true if the points in the neighbourhood of which these functions are unbounded are countable, provided the upper and lower functions possess Lebesgue or Harnack-Lebesgue integrals.†

As an immediate consequence we have, of course, the further result that the coefficients of such a series tend, in the first two cases, towards zero as unique limit, and that what may be called for convenience the integrated series, in which a_n/n takes the place of b_n and $-b_n/n$ takes the place of a_n, necessarily converges uniformly everywhere, the function to which it converges being, moreover, an integral of the same type—Lebesgue or Harnack-Lebesgue—as the coefficients.

The methods employed are, it is hardly necessary to say, based on the work of Riemann. All turns, however, on the use of a theorem stated and proved in a paper, recently presented to the London Mathematical Society, "On Semi-integrals and Oscillating Successions of Functions." My result clearly contains previously known ones as very particular cases, and the reasoning employed is in its essence a generalisation of that at present in existence. It is hoped that it will be found to be also more concisely arranged and more easily followed.

2. We begin by proving a theorem which gives us the necessary and sufficient condition that a trigonometrical series should be an ordinary Fourier series, using this expression to denote that the integrals involved in the Fourier form of the coefficients are absolutely convergent proper or improper Lebesgue integrals. By a known theorem this will be the case if the function which occurs in these coefficients possesses itself a Lebesgue integral.

* Exception must, as Professor Hobson points out, be made of the theorem due to F. Riesz that a trigonometrical series is a Fourier series, if that formed by the squares of the coefficients converges. This theorem is, however, conceived in a quite different order of ideas from the present paper, and will, as may be seen by the examples given in the text, often fail when the theorems here explained apply (see § 6).

† It should be remarked that the argument shews that the last condition may be dispensed with, provided we know that the integrated series converges to a continuous function; in particular it will be sufficient to know that at the points of the exceptional countable set the integrated series conyerges uniformly (see § 6).

Denoting the original trigonometrical series by

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx),$$

we shall refer to the series

$$\tfrac{1}{2}a_0x+\sum_{n=1}^{\infty}(a_n\sin nx-b_n\cos nx)/n$$

as the "integrated series."

We may, as will be seen, without loss of generality, suppose the series to have no constant term. In this case the term $\frac{1}{2}a_0x$ will, of course, be absent from the integrated series.

That the condition about to be given is sufficient as well as necessary must, I think, be known, though the only statement I have seen in this connection is that which asserts that the integrated series of a Fourier series converges to the integral of the function associated with the series. In this form the theorem is due to Fatou, who extended Lebesgue's previous result.*

THEOREM.—*The necessary and sufficient condition that a trigonometrical series*

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

should be an ordinary Fourier series—in other words, that the coefficients should be expressible in the Fourier form—is that the series

$$\tfrac{1}{2}a_0x+\sum_{n=1}^{\infty}(a_n\sin nx-b_n\cos nx)/n$$

should converge, throughout the whole closed interval $-\pi$ to π, to a function which is an integral.

In this case, the function which appears in the Fourier form of the coefficients a_n and b_n may be taken to be any one of the derivates of this integral.

Let us first prove that the condition is sufficient. Put

$$g(x)=\sum_{n=1}^{\infty}(a_n\sin nx-b_n\cos nx)/n\,; \qquad (1)$$

then, by the hypothesis made, $g(x)$ is an integral, since it differs from an integral by $\frac{1}{2}a_0x$. Hence $g(x)$ is continuous, of bounded variation, and periodic, and can therefore be completely expressed as a Fourier series.

* See Fatou, *loc. cit.*, p. 384. In Hobson's *Theory of Functions of a Real Variable*, only a special case of Lebesgue's original result is given, the restriction arising from the fact that the theorem of integration by parts is not used in its most general form.

Also, no two convergent trigonometrical series can represent the same function, so that the series (1) is the Fourier series representing $g(x)$. Hence

$$-b_n/n = \frac{1}{\pi}\int_{-\pi}^{+\pi} g(x)\cos nx\,dx.$$

Since $g(x)$ and $\sin nx$ are both integrals, we may integrate by parts and get

$$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} g'(x)\sin nx\,dx, \tag{2}$$

where $g'(x)$ denotes any one of the derivates of $g(x)$. Now, since $g(x)$ is an integral, it is the integral of any one of its derivates; and, since $g(x)$ is periodic, we get

$$\int_{-\pi}^{+\pi} g'(x)dx = 0. \tag{3}$$

Hence, denoting by $f(x)$ the sum of the integrated series, so that

$$f(x) = \tfrac{1}{2}a_0x + g(x),$$

we get

$$f'(x) = \tfrac{1}{2}a_0 + g'(x),$$

so that, by (3),

$$\frac{1}{\pi}\int_{-\pi}^{+\pi} f'(x)dx = a_0,$$

and, by (2),

$$\frac{1}{\pi}\int_{-\pi}^{+\pi} f'(x)\sin nx\,dx = b_n;$$

similarly

$$\frac{1}{\pi}\int_{-\pi}^{+\pi} f'(x)\cos nx\,dx = a_n.$$

(4)

The equations (4) prove the sufficiency of the conditions.

Next, suppose the given trigonometrical series to be a Fourier series and let the equations (4) hold, $f'(x)$ denoting the function which enters into the Fourier form of the coefficients, and which is accordingly assumed to be summable, and is therefore, except at a set of content zero, the differential coefficient of its integral, which will be denoted by $f(x)$.

Put

$$g(x) = f(x) - \tfrac{1}{2}a_0x;$$

then $g(x)$ is, like $f(x)$, an integral, and is periodic, since equation (3) holds, so that, $g(x)$ being the integral of $g'(x)$, $g(-\pi) = g(\pi)$.

We then have, by (4),

$$-b_n/n = -\frac{1}{\pi}\int_{-\pi}^{+\pi} g'(x)\sin nx\,dx/n,$$

which we may, as before, integrate by parts and get

$$-b_n/n = \frac{1}{\pi}\int_{-\pi}^{+\pi} g(x)\cos nx\,dx.$$

Similarly,

$$a_n/n = \frac{1}{\pi}\int_{-\pi}^{+\pi} g(x)\sin nx\,dx.$$

(5)

The equations (5) shew that the series on the right in equation (1) is the Fourier series of $g(x)$, the first term of which is zero, since $g(x)$ is periodic. But $g(x)$ is an integral and therefore has bounded variation, and therefore is expressible as its Fourier series. Thus equation (1) holds.

Hence the integrated series only differs from $g(x)$ by the addition of $\frac{1}{2}a_0x$, and is therefore also convergent and its sum is an integral. Thus the conditions are necessary as well as sufficient, which proves the theorem.

COR.—*If a trigonometrical series is a Fourier series, it remains a Fourier series if any finite number of terms be omitted; in particular, this is the case if we omit the constant term* $\frac{1}{2}a_0$.

3. In the previous article we have assumed the Lebesgue definition of integration, which requires the integrand to be absolutely integrable. We shall require, however, in the sequel, to consider Harnack integrals, or, more generally, Harnack-Lebesgue integrals, in the special case in which the Harnack points* form a countable set. We must then modify the theorem just given so as to admit the possibility of the integrals involved being Harnack-Lebesgue integrals of this type. It follows easily, by Cantor induction, that the theorem is true in this case. We have accordingly the following theorem.

THEOREM.—*The necessary and sufficient condition that a trigonometrical series should be the generalised Fourier series of a function having only a countable set of Harnack points is that the integrated series should, except at a countable set of points, converge throughout the*

* The definition of a Harnack integral depends on the existence of a certain set of points, which we may call Harnack points, of zero content. A Harnack point is a point such that the function considered does not possess an integral, proper or improper, but absolutely convergent, in Riemann's or Lebesgue's sense, in any interval containing the Harnack point. Such points form, of course, a closed set. If this closed set have positive content, the function cannot be said to have an integral at all. If, however, as in Harnack's case, it is a set of content zero, we can enclose it in a set of small intervals, of content say E, each interval containing at least one point of the set, so that these small intervals are finite in number. Over each of the complementary intervals the function considered has an integral. If the sum of these integrals has a unique finite limit, when the content E diminishes indefinitely, however the small intervals be drawn, the function is said to have that limit for Harnack integral, or for Harnack-Lebesgue integral, according as the integrals summed are Riemann or Lebesgue integrals.

In the case of a Harnack integral, the Harnack points are the points of infinite discontinuity of the function. In the case of a Harnack-Lebesgue integral, they are all points of infinite discontinuity, but there may be points of infinite discontinuity which are not Harnack points; indeed, the points of infinite discontinuity may fill up a whole interval, but the Harnack points are necessarily dense nowhere.

whole interval $(-\pi, \pi)$ *to a function* $f(x)$ *which is the Harnack-Lebesgue integral of a function having only a countable set of Harnack points.**

Moreover, the function which appears in the Fourier form of the coefficients of the series may be taken to be any one of the derivates of this integral.

To see that this is so, we remark first that a Harnack-Lebesgue integral necessarily possesses a differential coefficient, except at a set of points of content zero, since it is an ordinary Lebesgue integral in every black interval of the set of Harnack points. Also, in each black interval this differential coefficient is the integrand, except at a set of content zero, so that a Harnack-Lebesgue integral is the integral of its differential coefficient over the set of points at which this differential coefficient exists.

Again, the formula for integration by parts,

$$[uv]_a^x = \int_a^x uv'\,dx + \int_a^x vu'\,dx,$$

which holds whenever u and v are Lebesgue integrals, proper or improper, is easily seen to hold still in the case here required in which u is a Harnack-Lebesgue integral with only a countable set of Harnack points and v is a sine or cosine. In fact, if c and x be not separated by any Harnack point of the integrand of the integral u, we have, for example,

$$[u \sin nx]_c^x = n\int_c^x u \cos nx\,dx + \int_c^x u' \sin nx\,dx,$$

the integrals being Lebesgue integrals. But the left-hand side of this equality is continuous throughout the whole interval $(-\pi, \pi)$, and the former of the two integrals on the right has a unique limit when the point x moves up to the nearest Harnack point. Hence the same is true of the other integral, and therefore, by the definition of a Harnack-Lebesgue integral, the equality still holds when the point x is the Harnack point in question.

Considering two black intervals abutting at an isolated Harnack point, and adding the corresponding equations, we see that the formula for integration by parts holds unimpaired by the presence of such an isolated Harnack point in the interval of integration. Hence, by Cantor induction, the formula holds for any interval whatever.

It follows from the above that the proof given in § 2 holds, in our case, as it stands, except that, the boundedness of the variation of $g(x)$ at the Harnack-Lebesgue points being doubtful, we have to use the theorem that

* If the points π and $-\pi$ are Harnack-Lebesgue points, we must add the condition that $f(\pi) = f(-\pi) + a_0\pi$.

no two trigonometrical series can, except at a countable set of points, converge to the same function (see my "Note on Trigonometrical Series," 1908, *Messenger of Mathematics*, pp. 44–48, and F. Bernstein, *Zur Theorie der trigonometrischen Reihen*," 1908, Leipzig, Ber., pp. 325–33).

4. Before proceeding to the main theorem of the paper, we must first prove the following lemma.

LEMMA.—*If the series* $\sum_{n=1} (a_n \cos nx + b_n \sin nx)$ *is such that, throughout some interval the upper and lower functions are finite, then the coefficients* a_n *and* b_n *lie, as* n *increases, between finite upper and lower bounds.*

If we suppose that the interval in question contains the origin, the statement is obviously true of the a's, as is seen by putting $x = 0$. Suppose in this case that it is not true of the b's. Then we can find a set of values of n for which b_n, remaining always, if we please, of the same sign, increases without limit. From this set of values of n we can, by Fatou's* reasoning, pick out a set for which $\sin nx$ remains greater than a fixed positive quantity. Thus the series cannot oscillate finitely.

If the origin is not a point of the interval, we can move the origin so that it becomes an internal point x_0 of the interval. It then follows, by the first case, that, as n increases,

$$a_n \cos nx_0 + b_n \sin nx_0$$

and

$$a_n \sin nx_0 - b_n \cos nx_0,$$

lie between finite upper and lower bounds.

Squaring and adding, we see that $a_n^2 + b_n^2$ lies between finite upper and lower bounds, whence the theorem follows.

COR.—*Under the same circumstances* $(a_n \cos nx + b_n \sin nx)$ *is, throughout the interval* $-\pi$ *to* π, *a bounded function of the ensemble* (x, n).

5. We now state and prove the theorem in question.

THEOREM.—*If the trigonometrical series*

$$\tfrac{1}{2}a_0 + \Sigma(a_n \cos nx + b_n \sin nx) \qquad \text{(A)}$$

has its upper and lower functions bounded, then it is a Fourier series.

More generally, if the upper and lower functions are bounded, except in the neighbourhood of a countable set of points, the same is true, provided only these upper and lower functions possess either Lebesgue or Harnack-Lebesgue† integrals throughout the interval $-\pi$ to π.

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* *Loc. cit.*, pp. 395–397.

† It will be convenient to suppose that the extremities of the interval $(-\pi, \pi)$ are not Harnack points. Note also that in this case the series is a *generalised* Fourier series.

We first remark that it follows by the lemma that $a_n \cos nx + b_n \sin nx$ is a bounded function of the ensemble (x, n), for all values of x from $-\pi$ to π and all positive integral values of n. From this it follows that, if we write

$$F(x) = -\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)/n^2, \qquad \text{(B)}$$

the series (B) is uniformly convergent and $F(x)$ is continuous.

We consider first the case where the upper and lower functions of (A) are everywhere bounded. In this case

$$\{F(x+h)+F(x-h)-2F(x)\}/h^2, \quad \text{or, say, } G(x, h),$$

is, as is easily seen, by a theorem of du Bois Reymond's,* a bounded function of the ensemble (x, h). Hence, by a known theorem,† the upper and lower limits of its integral are themselves integrals. Thus, denoting the integral of $F(x)$, between the limits c and x, by $F_1(x)$, the upper and lower functions of the succession of functions

$$\{F_1(x+h)+F_1(x-h)-2F_1(x)\}/h^2, \quad \text{or, say, } G_1(x, h),$$

are integrals.

Since this last succession of functions is also bounded in its entirety, it follows in the same way that the upper and lower functions of the succession formed by their integrals are themselves integrals. In other words, denoting by $F_2(x)$ the integral of $F_1(x)$ between the limits c and x, the upper and lower functions of the succession of functions

$$\{F_2(x+h)+F_2(x+h)-2F_2(x)\}/h^2, \quad \text{or, say, } G_2(x, h)$$

are both integrals.

But $F_2(x)$ possesses a second differential coefficient, namely, $F(x)$; hence the last-mentioned upper and lower functions coincide with $F(x)$. Hence $F(x)$ is an integral, and has therefore, by a theorem of Lebesgue's, a finite differential coefficient, except at a set of content zero. Therefore $F_1(x)$ has, except at a set of content zero, a finite second differential coefficient.

It follows that the succession of functions $G_1(x, h)$ has, except at a set of values of x of content zero, a unique limit. But the upper and lower functions of this succession are both integrals, and therefore continuous; hence they must coincide everywhere. Thus the succession of functions $G_1(x, h)$ is a sequence and its limiting function is an integral.

* *Loc. cit.* Quoted in Hobson's *Theory of Functions of a Real Variable*, § 481, p. 732. The theorem states that the limits of $G(x, h)$ lie between two functions of the form $Lu(x) + Mu(x)$, where L and M are constants, and $u(x)$ and $l(x)$ are the upper and lower functions of the original series. By a theorem of Hölder's (*vide loc. cit.*, § 488, pp. 750, 751) the same is then true of $G(x, h)$ itself.

† W. H. Young, "On Semi-integrals and Oscillating Successions of Functions," § 16, *supra*, p. 286.

The unique limit of the sequence of functions $G_2(x, h)$ is, therefore, the integral of an integral; in other words, $F(x)$ is the integral of an integral, and therefore $F'(x)$ exists and is an integral.

Again, multiplying both sides of (B) by $\cos nx$ and $\sin nx$ respectively, and integrating in the usual way, we get

$$a_n/n^2 = -\frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos nx\,dx, \qquad b_n/n^2 = -\frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin nx\,dx.$$

Bearing in mind that $F(x)$ is periodic, we have at once, integrating by parts

$$a_n/n = \frac{1}{\pi}\int_{-\pi}^{\pi} F'(x)\sin nx\,dx, \quad -b_n/n = \frac{1}{\pi}\int_{-\pi}^{\pi} F'(x)\cos nx\,dx.$$

Hence, the integrated series of (A), omitting the term $\frac{1}{2}a_0x$, is the Fourier series of $F'(x)$, that is, of an integral, and therefore converges to $F'(x)$, that is, to an integral. Hence, by the theorem of § 2 the required result follows.

Passing now to the second case, we remark that $F''(x)$ still necessarily exists except at a set of content zero, since, by the first case, it exists in each black interval of the exceptional countable set, excepting a set of content zero in that black interval. Moreover, except at a set of content zero, it follows, by du Bois Reymond's theorem quoted above, that $F''(x)$ lies between two functions of the form $Lu(x)+Ml(x)$, where $l(x)$ and $u(x)$ are the lower and upper functions of the given trigonometrical series, that is, between two functions, each of which possesses an integral.

If these integrals are Lebesgue integrals, it follows that $F''(x)$ also possesses a Lebesgue integral in the whole interval. If these integrals are Harnack integrals, or Harnack-Lebesgue integrals, it follows equally that $F''(x)$ has a Harnack or Harnack-Lebesgue integral in the whole interval.* Hence the integral of $F''(x)$ has a unique limit as the upper limit of integration approaches an isolated point of the given countable set. $F'(x)$ therefore has a unique limit as x approaches such an isolated point. Thus $F'(x)$ exists and is continuous at each such isolated point. Hence, by Cantor induction, $F'(x)$ exists and is continuous throughout the whole interval $-\pi$ to π, being, moreover, the integral of $F''(x)$. Thus $F(x)$ is, throughout the whole of that interval, the integral of that integral. From this point the argument is the same as in the first case, using the theorem of § 3. Hence the theorem is true in all cases.

* For, if $h(x) \leqslant f(x) \leqslant g(x)$ everywhere, g and h both possess Harnack-Lebesgue integrals, the function $g-h$ is a positive function possessing a Harnack-Lebesgue integral, which is therefore a Lebesgue integral. Since $g-f$ is also a positive function, and is $\leqslant g-h$, it follows that $g-f$ has a Lebesgue integral. Hence, since g has a Harnack-Lebesgue integral, $f+(g-f)$, that is, f, has also a Harnack-Lebesgue integral.

6.* It should be noticed that we have really proved more than is involved in the statement of the theorem. We only use the integrability of the upper and lower functions in order to prove that at the exceptional points the integrated series converges to a continuous function. If we know the latter fact independently, we can dispense with the integrability of the upper and lower functions. This remark, though obvious, is important in the applications; I therefore state the result as a separate theorem.

THEOREM.—*If a trigonometrical series has its upper and lower functions bounded except in the neighbourhood of a countable set of points, and if throughout the whole interval* $(-\pi, \pi)$ *the integrated series converges to a continuous function* (*e.g., if it converges uniformly in the neighbourhood of the exceptional points*), *then the trigonometrical series is a Fourier series.*

As examples, we may take the series

$$\Sigma n^{-p} \sin nx, \quad \Sigma n^{-p} \cos nx, \quad (0 < p). \tag{1}$$

These series converge, in fact, boundedly, except in the neighbourhood of $x = 0$, since, with the same exception, the series

$$\Sigma \sin nx \quad \text{and} \quad \Sigma \cos nx$$

oscillate boundedly, and we may apply Abel's lemma. Further, the integrated series evidently converge uniformly everywhere. Hence the series (1) in question are Fourier series.

7. It is also worthy of remark that the methods here explained may be utilized to prove that certain trigonometrical series, though not the Fourier series of functions which possess either a Lebesgue or a Harnack-Lebesgue integral in the whole interval $(-\pi, \pi)$, are none the less Fourier series. Such series must, of course, have no constant term.

An interesting example is furnished by one of the two subsidiary series used in the preceding article, namely, the series whose general term is $\sin nx$. The integrated series here converges everywhere, except at $x = 0$, to a function which must be an integral, since the original series converges boundedly, except in the neighbourhood of $x = 0$. Denoting the sum of the integrated series by $F(x)$, we easily see both that the Fourier constants of $F(x)$ exist, and that we may, by means of integration by parts, transform them into n times the corresponding Fourier constants of $F'(x)$ with proper sign.

* This and the following articles were added November 20th.

In fact,
$$F(x) = \tfrac{1}{2}\log\frac{1}{2(1-\cos x)},$$
and has therefore an integral in the whole interval $(-\pi, \pi)$, since $\log(\sin\frac{1}{2}x)^2$ has. Hence the Fourier coefficients of $F(x)$, of course, exist. Further, $F(x)\sin nx$ has, as x approaches the limit zero, the unique limit zero, since $\log x \sin nx$ has. Thus we have

$$-n\int_0^\pi F(x)\cos nx\,dx$$
$$= -\operatorname*{Lt}_{e=0}\int_e^\pi F(x)n\cos nx\,dx = -\operatorname*{Lt}_{e=0}\left[F(x)\sin nx\right]_e^\pi + \operatorname*{Lt}_{e=0}\int_e^\pi F'(x)\sin nx\,dx,$$
and therefore
$$= \operatorname*{Lt}_{e=0}\int_e^\pi F'(x)\sin nx\,dx.$$

Similarly,
$$-n\int_{-\pi}^0 F(x)\cos nx\,dx = \operatorname*{Lt}_{e_1=0}\int_{-\pi}^{e_1} F'(x)\sin nx\,dx.$$

Hence, adding, we see that $\int_{-\pi}^{\pi} F'(x)\sin nx\,dx$ exists and is equal to
$$-n\int_{-\pi}^{\pi} F(x)\cos nx\,dx.$$

This proves that the series whose general term is $\sin nx$ is a Fourier series, which is the result stated above. It is accordingly the Fourier series of $F'(x)$, that is, of $\frac{1}{2}\cot\frac{1}{2}x$, a fact which is easy to verify directly by shewing, *e.g.*, by induction, that the Fourier constants of this function are all unity. If, on the other hand, we attempt to carry out the same process with the series whose general term is $\cos nx$, we shall find that the method will fail. For, although the series converges boundedly, except in the neighbourhood of the point $x = 0$, the process of integration by parts will be found no longer to lead to the desired result.

8. In conclusion, it is desirable to emphasize the practical importance of a knowledge that a given trigonometrical series is a series of Fourier. Whenever this is the case, the series may be treated in many respects as if it converged, even when, as happens of course in the general case, the points at which there is oscillation or divergence form a set of positive content. We may, in fact, multiply such a series by any function of bounded variation and integrate term-by-term with the same result as if the series converged to the function of which it is the Fourier series.*

* See my forthcoming paper "On the Integration of Fourier Series," *infra*, p. 449.

Supplementary Note. Added November 18th, 1910. — In § 3 I contented myself with extending the theorem of § 2, as far as was necessary for the immediate purpose in hand, namely, of proving the theorem of § 5. It is worth noting, however, that we can proceed still further than would appear from that article. This depends on the fact that the theorem of integration by parts can be shewn to be true when the Harnack points of one of the functions to be integrated are not merely countable, but have the most general distribution possible consistent with the existence of the integral, when, namely, they form any closed set of content zero. It thus appears that the integrated series of a generalised Fourier series is the Fourier series of the Harnack-Lebesgue integral of the function associated with the original series. As this integral is not of bounded variation, the integrated series will not in general converge at the Harnack points, although it converges uniformly everywhere when summed in the Cesàro manner.

The importance of this further theorem, which I have needed in my later researches into the theory of Fourier series, seemed to render it desirable to refer to the matter here. I append the proof of the theorem of integration by parts in the new form here required.

It will be remarked that I have here used a principle which I have been able to dispense with in § 3, namely, that of the uniformity of approach of the sequence defining a Harnack-Lebesgue integral to its limit.* On the other hand, even the very simple form of Cantor induction used in the main body of the paper has been avoided.

Theorem.—*If F is the Harnack-Lebesgue integral of $f(x)$ and G is the Lebesgue integral of $g(x)$, then*

$$[FG]_c^z = \int_c^z Fg\,dx + \int_c^z Gf\,dx.$$

Enclose the Harnack points of the function $f(x)$ in the usual manner in a finite number of intervals, each containing at least one Harnack point, and let d_n denote this set of intervals. Let f_n denote the function got from f by changing its value at every point of the intervals d_n to zero, and let F_n denote the integral of f_n; then F_n is a Lebesgue integral. Hence

$$[F_n G]_c^z = \int_c^z F_n g\,dx + \int_c^z Gf_n\,dx. \qquad (1)$$

* See Hobson's treatise, p. 383.

Let the content of d_n, as n increases, approach zero as limit, then $F_n(x)$ approaches $F(x)$ uniformly as limit. Hence, since $g(x)$ is summable,

$$\operatorname*{Lt}_{n=\infty} \int_c^z F_n(x)\, g(x)\, dx = \int_c^z \operatorname*{Lt}_{n=\infty} F_n(x)\, g(x)\, dx = \int_c^z F(x)\, g(x)\, dx.$$

Hence the first term of the right-hand side of (1) has a unique limit, and therefore the same is true of the second term, as plainly the left-hand side has the limit $[FG]_c^z$. But, if the limit of $\int_c^z f_n(x)\, G(x)\, dx$ exists, it is, by definition, $\int_c^z f(x)\, G(x)\, dx$. Hence we have the required equation

$$[FG]_c^z = \int_c^z F(x)\, g(x)\, dx + \int_c^z G(x)\, f(x)\, dx.$$

COR.—*If $f(x)$ has a Harnack-Lebesgue integral and $G(x)$ is an integral, then $f(x)\, G(x)$ has a Harnack-Lebesgue integral.*

This is, of course, a special case of a known theorem.

Since two Fourier series cannot agree except at a set of content zero without coinciding, the reasoning of § 2 shews that, if the integrated series of a trigonometrical series is the Fourier series of a Harnack-Lebesgue integral $f(x)$, the points π and $-\pi$ not being Harnack-Lebesgue points, or else $f(\pi) = f(-\pi) + a_0\pi$, the trigonometrical series itself is a Fourier series of the generalised or Harnack-Lebesgue type. This is the converse of what was stated above: we have therefore the following *necessary and sufficient condition that a trigonometrical series should be a generalised Fourier series:—the integrated series must be the Fourier series of a Harnack-Lebesgue integral $f(x)$, π and $-\pi$ not being Harnack-Lebesgue points, or else $f(\pi) = f(-\pi) + a_0\pi$.*

THE CONVERSE OF ABEL'S THEOREM ON POWER SERIES

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Introduction.

Abel's theorem states that if $\sum_0^\infty a_n$ is convergent, then $\lim \sum_0^\infty a_n x^n$ exists as $x \to 1$ by real values, and is equal to Σa_n. The converse theorem, however, that the existence of $\lim_{x \to 1} \Sigma a_n x^n$ implies the convergence of Σa_n, is very far from being true; for example, either the Cesàro or the Borel summability of Σa_n suffices for the existence of Abel's limit. It is known, however, that the existence of this limit, *combined with certain conditions satisfied by the a's*, does imply the convergence of Σa_n. Three such sets of conditions, for example, are:

(*a*)† the a's are all positive;

(*b*) the order of a_n has a certain upper limit;

(*c*)‡ the function $\Sigma a_n x^n$ is regular at the point $x = 1$ and $a_n \to 0$.

In the present paper we are concerned with the problems arising out of case (*b*), where the only additional restriction on the a's is an upper limit to the order of a_n. The theorem of this case is due to M. Tauber.§ The result is remarkable and apparently paradoxical in view of Abel's theorem, for it may be expressed roughly by saying that if Σa_n is not actually convergent, then the more nearly convergent it is, the more un-

* The paper has been modified considerably from the form in which it was first communicated. It then consisted of Theorems (A), (B), and (C), with the index λ_n limited to the form n^k, and the note in § 10. An introduction was added, and the title changed, at the suggestion of the referees. I am indebted to Mr. G. H. Hardy for several important additions, and also for directing my attention to the subject in the first place. The application of Tauber's theorem to finitely oscillating power series, and the distinction in this respect between Tauber's case and that considered here are due to him, as also is the application to Fourier's series in § 9.

† Pringsheim, *Münchener Sitzungsberichte*, Bd. 30, 1900, p. 37. The result, however, was practically proved, although not explicitly stated, by Abel.

‡ Fatou, *Thèse*, Stockholm, 1906, p. 389.

§ *Monatshefte f. Math. u. Phys.*, Bd. 8, 1897, p. 273. Pringsheim (*Münchener Sitzungsberichte*, Bd. 31, 1901, p. 507) has generalised the theorem by showing that Σa_n converges if Abel's limit exists as $x \to 1$ along any path lying within any pair of straight lines which cut the circle at finite angles, and if $(a_1 + 2a_2 + \ldots + na_n)/n \to 0$.

likely is it that $\Sigma a_n x^n$ should exist.* M. Tauber's actual result is that the existence of $\lim_{x\to 1} \Sigma a_n x^n$, and the relation $na_n \to 0$, together involve the convergence of Σa_n. The proof of this theorem is as follows.†

We have $$\left| \sum_0^{\nu-1} a_n(1-x^n) \right| < (1-x) \sum_0^{\nu-1} n|a_n|,$$

because $$(1-x^n)/(1-x) = 1+x+x^2+\ldots+x^{n-1} \leqslant n;$$

also, if H_ν is the upper limit to $n|a_n|$, $(n+1)|a_{n+1}|$, ... to ∞, we have

$$\left| \sum_\nu^\infty a_n x^n \right| < H_\nu \nu^{-1} x^\nu (1+x+x^2+\ldots) < H_\nu \nu^{-1} x^\nu/(1-x).$$

Take $x = 1-\nu^{-1}$; we then have

$$\left| \sum_0^{\nu-1} a_n - \sum_0^\infty a_n x^n \right| < \nu^{-1} \sum_0^{\nu-1} n|a_n| + H_\nu. \qquad (1)$$

As $\nu \to \infty$, each of the terms on the right tends to zero; and so

$$\lim_{\nu\to\infty} \sum_0^{\nu-1} a_n = \lim_{x\to 1} \sum_0^\infty a_n x^n,$$

which proves the theorem.

If the restriction in this theorem on the order of $|a_n|$ is the minimum possible, we ought to be able to find non-convergent series for which $\overline{\lim}\, na_n > 0$, but is finite, and for which Abel's limit exists. Now the easiest means of constructing a non-convergent series for which Abel's limit exists is to take a series summable by Cesàro's first mean. Mr. Hardy‡ has shown, however, that *no* non-convergent series for which $|na_n| < K$ can be summable by any one of Cesàro's means. The method therefore fails to provide an instance of the desired kind, and we are naturally led, as Mr. Hardy himself remarks, to ask whether Tauber's theorem also does not hold under the wider condition $|na_n| < K$. That it does is proved in Theorem (B) of the present paper.

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* Cp. Hardy, "On Slowly Oscillating Series," *Proc. London Math. Soc.*, Ser. 2, Vol. 8, p. 315 (this paper will be referred to as "O.S."), where an explanation of the corresponding paradox relating to summability will be found.

† I have taken the proof in Bromwich's *Infinite Series*, p. 251, adapting it to the case when $x \to 1$ by real values.

‡ "O.S.," p. 308.

The proof of Tauber's theorem given above proves more than is actually stated. It follows immediately from (1) that, if $\Sigma a_n x^n$ oscillates finitely as $x \to 1$, then the limits of oscillation, as $n \to \infty$, of Σa_n are the same as the limits of oscillation of $\Sigma a_n x^n$. Now it is a somewhat remarkable fact that with the condition $|na_n| < K$ this result no longer holds. To see this it is sufficient to consider the series

$$\Sigma \mathfrak{R} n^{-1-\alpha\iota}, \qquad \Sigma \mathfrak{R} n^{-1-\alpha\iota} x^n.$$

The well known asymptotic formula

$$\sum_1^n n^{-1-\alpha\iota} \sim n^{-\alpha\iota}/\alpha\iota$$

shows that the limits of oscillation of Σa_n are $\pm \alpha^{-1}$. On the other hand, as $x \to 1$, we have*

$$\sum_1^\infty n^{-1-\alpha\iota} x^n \sim \Gamma(\alpha\iota)\,[\log(1/x)]^{\alpha\iota}.$$

The limits of oscillation of the *function* $\Sigma a_n x^n$ are therefore

$$\pm |\Gamma(\alpha\iota)| \quad \text{or} \quad \pm \alpha^{-1} \sqrt{(\pi\alpha \operatorname{cosech} \pi\alpha)},$$

and are different from those of the *series*.

It follows from this result that it is *not* always possible to find a relation between x and ν for which the left-hand side of (1) tends to zero. It is to be expected, therefore, that the extended theorem will require a new and more complicated proof.

The series for which $|na_n| < K$ are much more interesting than those for which $na_n \to 0$; the former class includes, for example, all Fourier's series which satisfy Dirichlet's conditions. Mr. Hardy has shown† that, in virtue of his theorem that a series for which $|na_n| < K$ is convergent if it is summable by Cesàro's method, Fourier's theorem with Dirichlet's conditions becomes an immediate corollary of Fejér's theorem that the Fourier's series for any continuous function is summable by Cesàro's first mean. In the same way our extension of Tauber's theorem exhibits Fourier's theorem as a corollary of the result known under the name of "Poisson's Integral," thereby completing Poisson's attempted proof of Fourier's theorem.

It is shown in Theorem (C) of this paper that the condition $|na_n| < K$ is the minimum possible restriction of its kind; *i.e.*, that, given any

* Lindelöf, *Le Calcul des Résidus*, p. 139.

† "O.S.," p. 308.

function $\phi(n)$ tending to infinity, there exists a non-convergent series for which Abel's limit exists, while $|na_n| < \phi(n)$.

M. Landau has proved an analogue of Tauber's theorem for Dirichlet's series. His result is that, if $\Sigma\mu_n$ is a divergent series of positive terms, the existence of $\lim_{x\to 0} \sum_1^\infty a_n e^{-\lambda_n x}$ and $\lambda_n a_n/\mu_n \to 0$, where

$$\lambda_n = \mu_1+\mu_2+\ldots+\mu_n,$$

together imply the convergence of Σa_n. (Tauber's theorem is the special case $\mu_n = 1$.) This theorem may be extended in the same way as Tauber's: the general index λ_n will, in fact, be considered throughout the proof of Theorem (B). No additional complication results from this procedure; indeed, the general problem suggests a proof considerably simpler in some details than my original proof for the case of the index n.

Theorem (C) also holds with the general index, with, however, one very important reservation. When some unimportant conditions are satisfied by the μ's, when $\mu_n/\lambda_n \to 0$ (a restriction roughly equivalent to $\lambda_n < e^{\epsilon n}$, where ϵ is arbitrarily small), and when $\phi(n) \to \infty$, non-convergent series Σa_n exist such that

$$\Sigma a_n e^{-x\lambda_n} \to s \quad \text{and} \quad |\lambda_n a_n/\mu_n| < \phi(n).$$

But when the order of λ_n is comparable with or greater than that of $e^{\rho n}$, the theorem breaks down entirely. I have not completely solved the problem of these high indices, but I feel practically certain that the existence of $\lim \Sigma a_n e^{-x\lambda_n}$, by itself, implies the convergence of Σa_n. For the present, therefore, I have excluded these cases from consideration in connexion with Theorem (C). In Theorem (B) they are not exceptions, but they require a modified proof, and in view of the fact that the relations of that theorem constitute only a small part of the whole truth about them I have omitted them there also.

The paper concludes with some applications of Theorems (B) and (C), including that mentioned above to Fourier's series, and with a note on the converse of Abel's theorem, with summability of Σa_n in the place of convergence.

1. In the proof of Theorem (B) the following theorem is fundamental.

THEOREM (A).—*If, as* $x \to \infty$ *by real values,* $\phi(x) \to s$, *and every derivate of* $\phi(x)$ *is finite, then every derivate of* $\phi(x)$ *tends to zero.*

We may assume, without loss of generality, that $\phi(x)$ is real.

Suppose that $\phi'(x)$ oscillates. Then there exists a positive constant h such that, for arbitrarily large values x_n of x, $|\phi'(x)| > h$. Let $\phi'(x_n)$ be, say, positive. Then, since $|\phi''(x)| < K$,

$$\phi'(x)-\phi'(x_n) = \int_{x_n}^{x} \phi''(x)\,dx > -K|x-x_n|,$$

and therefore $$\phi'(x) > \tfrac{1}{2}h,$$

when $$|x-x_n| < \tfrac{1}{2}hK^{-1} = c.$$

Then $$\int_{x_n-c}^{x_n+c} \phi'(x)\,dx > ch.$$

This is incompatible with $\int^{\infty} \phi'(x)\,dx = s$, and it follows that $\phi'(x) \to 0$.

The argument may evidently be repeated with ϕ', ϕ'', ... successively in the place of ϕ, and we have

$$\phi^{(r)}(x) \to 0 \quad (r \geqslant 1).$$

COROLLARY.—*If, as* $y \to 0$ *by positive values,*

$$\psi(y) \to s \quad \textit{and} \quad |y^r \psi^{(r)}(y)| \textit{ is finite} \quad (r \geqslant 1),$$

then $$y^r \psi^{(r)}(y) \to 0 \quad (r \geqslant 1).$$

For, writing $$y = e^{-x}, \qquad \psi(y) = \phi(x),$$

we have $$(-y)^r \psi^{(r)}(y) = D(D-1)\ldots(D-r+1)\,\phi(x) \quad (D = d/dx). \quad (1)$$

By induction from (1) we have $|\phi^{(r)}(x)| < K$; therefore, by the theorem, $\phi^{(r)}(x) \to 0$, and therefore, from (1),

$$y^r \psi^r(y) \to 0.$$

2. THEOREM (B).—*If* $\Sigma\mu_n$ *is a series of positive terms such that*

$$\lambda_n = \mu_1+\mu_2+\ldots+\mu_n \to \infty, \qquad \mu_n/\lambda_n \to 0,$$

then Σa_n *is convergent, provided*

$$\sum_1^{\infty} a_n e^{-x\lambda_n} \to s,$$

and $$|a_n| < K\mu_n/\lambda_n.$$

We may assume without loss of generality that a_n is real.

The first stage in the proof is to establish the following result:—

$$x^{r+1}\sum_1^\infty s_n\int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt \to r!\,s \quad (r \geqslant 1), \tag{2}$$

where $$s_n = a_1+a_2+\ldots+a_n.$$

We observe once for all that the inequality for a_n involves the absolute convergence of $\Sigma a_n\lambda_n^p e^{-x\lambda_n}$, when $x > 0$. For

$$\begin{aligned} |\, a_n\lambda_n^p e^{-x\lambda_n}| &< K\mu_n\lambda_n^{p-1}e^{-x\lambda_n} \\ &< K\mu_n\lambda_n^{-2} \quad (\text{since } \lambda_n\to\infty) \\ &< K\int_{\lambda_{n-1}}^{\lambda_n} x^{-2}\,dx, \end{aligned}$$

and the absolute convergence of the series follows immediately from the convergence of $\int^\infty x^{-2}\,dx$. From this result, and the inequality $|s_n| < K\lambda_n$, may be deduced without difficulty the legitimacy of the rearrangements and differentiations of the series $\Sigma a_n e^{-x\lambda_n}$ which occur in the sequel.

We note also that $\mu_n/\lambda_n \to 0$ involves

$$\lambda_{n+1}/\lambda_n \to 1 \quad \text{and} \quad \lambda_{n+1} < K\lambda_n.$$

Let $$\psi(x) = \sum_1^\infty a_n e^{-x\lambda_n} = \Sigma s_n(e^{-x\lambda_n}-e^{-x\lambda_{n+1}}) = x\,\Sigma s_n\int_{\lambda_n}^{\lambda_{n+1}} e^{-xt}\,dt. \tag{3}$$

Then, since $|a_n| < K\mu_n/\lambda_n$,

$$\begin{aligned} |\, x^r\psi^{(r)}(x)| &= x^r\,|\,\Sigma a_n\lambda_n^r e^{-x\lambda_n}| < Kx^r\Sigma\mu_n\lambda_n^{r-1}e^{-x\lambda_n} \\ &< Kx^r\Sigma\mu_n(K\lambda_{n-1})^{r-1}e^{-x\lambda_n} \\ &< Kx^r\Sigma\int_{\lambda_{n-1}}^{\lambda_n} t^{r-1}e^{-xt}\,dt < Kx^r\int_0^\infty t^{r-1}e^{-xt}\,dt \\ &< K. \end{aligned}$$

Hence, by means of the corollary to Theorem (A),

$$x^r\psi^{(r)}(x) \to 0.$$

With the second form of $\psi(x)$ in (3) this gives

$$(-)^r\left[x^{r+1}\Sigma s_n\int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt - rx^r\Sigma s_n\int_{\lambda_n}^{\lambda_{n+1}} t^{r-1}e^{-xt}\,dt\right] \to 0,$$

whence $$V_r \equiv x^{r+1}\Sigma s_n \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt = rV_{r-1}+\epsilon_x = \ldots \quad (r \geqslant 1)$$
$$= r!\,V_0+\epsilon_x \to r!\,s,$$

the desired result (2).

3. We now write $s_n = s+\sigma_n$; then, since

$$x^{r+1}\Sigma \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt = x^{r+1}\int_{\lambda_1}^{\infty} t^r e^{-xt}\,dt = \int_{\lambda_1 x}^{\infty} y^r e^{-y}\,dy \to r!,$$

we have $$x^{r+1}\Sigma\sigma_n \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt \to 0 \quad (r \geqslant 1). \tag{4}$$

The convergence of Σa_n is equivalent to $\sigma_n \to 0$; the proof of our theorem consists in showing that (4) is false if σ_n does not $\to 0$. What we shall actually prove concerning (4) is that, if $|\sigma_n|$ exceeds a positive h for arbitrarily large values of n, then, when r exceeds a certain number depending on h,

$$\left| x^{r+1}\Sigma\sigma_n \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt \right| > K^{-1}$$

for a certain set of values of x tending to 0.

4. If σ_n does not $\to 0$, there exists an $h > 0$ such that $|\sigma_n| > h$ for an infinite number of values of n. Let m be such a value: then, if $n > m$,

$$|\sigma_n-\sigma_m| \leqslant |a_{m+1}|+|a_{m+2}|+\ldots+|a_n| < K\sum_{m+1}^{n} \mu_n\lambda_n^{-1}$$
$$< K\sum_{m+1}^{n}\int_{\lambda_{n-1}}^{\lambda_n} x^{-1}\,dx < K\log(\lambda_n/\lambda_m).$$

Similarly, if $m > n$, $\quad |\sigma_n-\sigma_m| < K\log(\lambda_m/\lambda_n)$.

Consequently

$$|F(x)| \equiv \left| x^{r+1}\sum_1^{\infty}\sigma_n\int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt \right|$$
$$\geqslant \left| x^{r+1}\sum_1^{\infty}\sigma_m\int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt \right| - \left| x^{r+1}\Sigma(\sigma_n-\sigma_m)\int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt \right|$$
$$> |\sigma_m|\, x^{r+1}\sum_1^{\infty}\int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt - Kx^{r+1}\sum_{m+1}^{\infty}\log(\lambda_n/\lambda_m)\int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt$$
$$-Kx^{r+1}\sum^{m-1}\log(\lambda_m/\lambda_n)\int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt \tag{5}$$

We choose $x = r/\lambda_m$, and consider the limits of each of the terms on the right-hand side of (5), as $m \to \infty$, *i.e.*, as $x \to 0$ through a certain set of values. (m, of course, is always an integer of a special kind.)

In the first of these terms

$$\lim_{x\to 0} x^{r+1} \sum_1^\infty \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt = r!,$$

and hence
$$\lim_{x\to 0} |\sigma_m|\, x^{r+1} \sum_1^\infty \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt \geqslant hr!. \tag{6a}$$

In the second term,

$$\log(\lambda_n/\lambda_m) \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt < \int_{\lambda_n}^{\lambda_{n+1}} \log(t/\lambda_m)\, t^r e^{-xt}\,dt,$$

and therefore

$$\overline{\lim_{m\to\infty}}\; x^{r+1} \sum_{m+1}^\infty \log(\lambda_n/\lambda_m) \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt$$

$$\leqslant \overline{\lim_{m\to\infty}}\; x^{r+1} \int_{\lambda_m}^\infty \log(t/\lambda_m)\, t^r e^{-xt}\,dt \leqslant \int_r^\infty \log(u/r)\, u^r e^{-u}\,du, \tag{6b}$$

where we have written $t = x^{-1}u = \lambda_m u r^{-1}$.

In the third term, since $\lambda_{n+1}/\lambda_n \to 1$,

$$\log(\lambda_m/\lambda_n) \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt = \log(\lambda_m/\lambda_{n+1}) \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt + \epsilon_n \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt,$$

where $\epsilon_n \to 0$ as $n \to \infty$,

$$< \int_{\lambda_n}^{\lambda_{n+1}} \log(\lambda_m/t)\, t^r e^{-xt}\,dt + \epsilon_n \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt,$$

so that

$$\overline{\lim_{m\to\infty}}\; x^{r+1} \sum_1^{m-1} \log(\lambda_m/\lambda_n) \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt$$

$$\leqslant \overline{\lim_{m\to\infty}} \int_0^{\lambda_m} \log(\lambda_m/t)\, t^r e^{-xt}\,dt + \overline{\lim_{m\to\infty}} \sum_1^{m-1} \epsilon_n x^{r+1} \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt.$$

If we write, as before, $t = ux^{-1} = \lambda_m u r^{-1}$,

the first term becomes $\int_0^r \log(r/u)\, u^r e^{-u}\,du.$

In the second term let ν be the value of n after which $|\epsilon_n| < \epsilon$. Then

$$\overline{\lim_{m\to\infty}} \sum_1^{m-1} \epsilon_n x^{r+1} \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt$$

$$< \overline{\lim_{x\to 1}} \sum_1^{\nu} Kx^{r+1} \int_{\lambda_n}^{\lambda_{n+1}} t\ e^{-xt}\,dt + \overline{\lim_{m\to\infty}} \sum_{\nu+1}^{m-1} \epsilon x^{r+1} \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt$$

$$< K \overline{\lim_{x\to 0}} \int_0^{x\lambda_{\nu+1}} u^r e^{-u}\,du + \epsilon \int_0^{\infty} u^r e^{-u}\,du \qquad (xt = u)$$

$$< 0 + \epsilon r!,$$

and therefore $= 0$,

since ϵ is arbitrary. Hence

$$\overline{\lim_{m\to\infty}}\ x^{r+1} \sum_1^{m-1} \log(\lambda_m/\lambda_n) \int_{\lambda_n}^{\lambda_{n+1}} t^r e^{-xt}\,dt \leqslant \int_0^1 \log(r/u)\, u^r e^{-u}\,du. \qquad (6c)$$

From (5) and (6*a*, *b*, *c*), we have

$$\overline{\lim_{x\to 0}}\,|F(x)| \geqslant hr! - K\left[\int_r^{\infty} \log(u/r)\, u^r e^{-u}\,du + \int_0^r \log(r/u)\, u^r e^{-u}\,du\right], \qquad (7)$$

in which the K is the K of (5), *and is independent of* r.

5. We shall now show that

$$I_1 = \int_0^r \log(r/u)\, u^r e^{-u}\,du < Kr^r e^{-r}, \qquad (8)$$

$$I_2 = \int_r^{\infty} \log(u/r)\, u^r e^{-u}\,du < Kr^r e^{-r}, \qquad (9)$$

where, as also in what follows, K denotes a constant independent of r.

Writing $u = r(1-z)$, we have

$$I_1 = r^{r+1} e^{-r} \int_0^1 [e^z(1-z)]^r \log[(1-z)^{-1}]\,dz. \qquad (10)$$

As z increases from 0 to 1, $e^z(1-z)$ decreases steadily from e to 0. Let c be the value of z for which

$$e^z(1-z) = e^{-1}.$$

Then, if $z \geqslant c$, $\qquad e^z(1-z) \leqslant e^{-1}$,

and if $\quad z \leqslant c, \quad e^z(1-z) < e^{-\frac{1}{2}z^2}$, and $\log[(1-z)^{-1}] < Kz$.

Therefore $\int_0^1 [e^z(1-z)]^r \log[(1-z)^{-1}]\,dz$

$$< \int_0^c e^{-\frac{1}{2}rz^2} Kz\,dz + e^{-r}\int_c^1 \log[(1-z)^{-1}]\,dz$$

$$< K\int_0^\infty e^{-\frac{1}{2}rz^2} z\,dz + Ke^{-r} < Kr^{-1} + Ke^{-r} < Kr^{-1},$$

and (8) follows from (10).

In the integral I_2 we write $u = r(1+z)$. Then

$$I_2 = r^{r+1}e^{-r}\int_0^\infty [(1+z)e^{-z}]^r \log(1+z)dz < Kr^{r+1}e^{-r}\int_0^\infty [(1+z)e^{-z}]^r\, z\,dz. \quad (11)$$

Now, if $z \leqslant \frac{1}{2}$, $\qquad (1-z)e^{-z} < e^{-\frac{1}{3}z^2}$,

[since $\qquad \log(1+z) < z - \frac{1}{2}z^2 + \frac{1}{3}z^3 < z - \frac{1}{3}z^2$],

and, if $z \geqslant \frac{1}{2}$, $\qquad (1+z)e^{-z} < e^{-\frac{1}{6}z}$,

[since $-\frac{5}{6}z + \log(1+z)$ decreases as z increases, and is negative when $z = \frac{1}{2}$]. Therefore

$$\int_0^\infty [(1+z)e^{-z}]^r z\,dz < \int_0^{\frac{1}{2}} e^{-\frac{1}{3}rz^2} z\,dz + \int_{\frac{1}{2}}^\infty e^{-\frac{1}{6}rz} z\,dz$$

$$< \int_0^\infty (e^{-\frac{1}{3}rz^2} + e^{-\frac{1}{6}rz})\, z\,dz < Kr^{-1} + Kr^{-2} < Kr^{-1},$$

and (9) follows from (11).

6. From (7), (8), and (9),

$$\overline{\lim_{x\to 1}}\,|F(x)| > hr! - Ke^{-r}r^r,$$

Now, by Stirling's theorem,

$$hr!/(Ke^{-r}r^r) \sim \sqrt{(2\pi)}\,hK^{-1}\,r^{\frac{1}{2}}.$$

Hence $\overline{\lim_{x\to 1}}\,|F(x)|$ is positive for sufficiently large values of r, and our theorem is proved.

7. We proceed now to the proof that the restriction on a_n is the minimum possible. Mr. Hardy has shown* that $\lim \Sigma a_n e^{-\lambda_n x}$ exists, if

$$\lim_{n\to\infty} s_n e^{-\lambda_n x} = 0,$$

when $x > 0$, and if Σa_n is summable $R(1, \mu_n)$, *i.e.*, if

$$\lim (\mu_1 s_1 + \mu_2 s_2 + \ldots + \mu_n s_n)/\lambda_n$$

exists.† To establish our result it is therefore sufficient to show that, given any function $\phi(n)$ tending to infinity, we can find a non-convergent series Σa_n which is summable $R(1, \mu_n)$, and for which

$$\lim_{n\to\infty} s_n e^{-\lambda_n x} = 0 \quad \text{and} \quad |\lambda_n a_n| < \phi(n)\,\mu_n.$$

This result requires a new condition, of an unimportant nature, to be satisfied by the μ's.

THEOREM (C).—*If*

$$\nu_n = \mu_n/\lambda_n \to 0, \quad \nu_{n-1}/\nu_n < K, \quad \textit{and} \quad \lambda_n = \mu_1 + \mu_2 + \ldots + \mu_n \to \infty,$$

then, given any function $\phi(n)$ tending to infinity, a finitely oscillating series Σa_n exists such that

$$\lim_{n\to\infty} (\mu_1 s_1 + \mu_2 s_2 + \ldots + \mu_n s_n)/\lambda_n \quad \textit{and} \quad \lim_{x\to 0} \Sigma a_n e^{-x\lambda_n}$$

exist, while $$|\lambda_n a_n| < \mu_n \phi(n).$$

Let $$\sigma_n = (\mu_1 s_1 + \mu_2 s_2 + \ldots + \mu_n s_n)/\lambda_n,$$

so that $$s_n = (\lambda_n \sigma_n - \lambda_{n-1}\sigma_{n-1})/\mu_n = \Delta\sigma_n/\nu_n + \sigma_{n-1},$$

where we write $\Delta f(n)$ for $f(n) - f(n-1)$.

We shall define the series Σa_n by means of the equation

$$\Delta\sigma_n/\nu_n = \Delta e^{i\psi_n}/\Delta\psi_n = f_n,$$

where ψ_n is a real function of n which will be chosen later.

Then $$\sigma_n = \sum^{n} \nu_n f_n, \qquad e^{i\psi_n} = \Sigma \Delta\psi_n f_n.$$

We shall suppose that $\psi_n \to \infty$, $\Delta\psi_n \to 0$. Then $e^{i\psi_n}$ and hence $\sum^{n} \Delta\psi_n f_n$ oscillate finitely. If now $\Delta\psi_n/\nu_n \to \infty$ steadily, it follows by Abel's test for convergence that $\Sigma \nu_n f_n$ is convergent, *i.e.*, that

$$\sigma_n = \sum^{n} \nu_n f_n \to \text{a definite limit},$$

so that Σa_n is summable $R(1, \mu_n)$.

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* "O.S.," p. 311.

† This generalised method of summation is due to Riesz, *Comptes Rendus*, July 5th, 1909. Cesàro's first mean has 1 in place of μ_n: if this mean exists we say that Σa_n is summable $(C1)$. The notation (Cr), $R(r, \mu_n)$ was introduced by Hardy.

Again, $\therefore \quad f_n = \Delta e^{\iota\psi_n}/\Delta\psi_n = e^{\iota\psi_n}(1-e^{-\iota\Delta\psi_n})/\Delta\psi_n$

$$\sim \iota e^{\iota\psi_n} \text{ (since } \Delta\psi_n \to 0).$$

Hence $s_n = f_n + \sigma_{n-1}$ oscillates finitely, since f_n does and $\sigma_{n-1} \to$ a limit.

Our series is now finitely oscillating and summable $R(1, \mu_n)$: we proceed to show that, with further assumptions as to ψ_n, we have also

$$|a_n| < K\phi(n)\nu_n.$$

Now
$$\begin{aligned} a_n = \Delta s_n &= \Delta f_n + \Delta\sigma_{n-1} \\ &= \Delta[e^{\iota\psi_n}(1-e^{-\iota\Delta\psi_n})/\Delta\psi_n] + \nu_n f_n \\ &= \Delta[\iota e^{\iota\psi_n} + O(\Delta\psi_n)] + O(\nu_n)^* \\ &= O(\Delta\psi_n) + O(\Delta\psi_{n-1}) + O(\nu_n). \end{aligned} \tag{12}$$

We can construct a function $\phi_1(n)$ tending steadily to infinity, and such that $\phi_1(n) < \phi(n)$, for example by choosing for $\phi_1(n)$ the minimum value of $\phi(m)$ for $m \geqslant n$. Suppose now that

$$\Delta\psi_n < \phi_1(n)\nu_n.$$

Then, from (12), $|a_n| < K\phi_1(n)\nu_n + K\phi_1(n-1)\nu_{n-1} + K\nu_n$

$$< K_1\phi_1(n)\nu_n \quad (\text{since } \nu_{n-1}/\nu_n < K).$$

Collecting our results we see that Σa_n is summable $R(1, \mu_n)$ and finitely oscillating, while

$$|a_n| < K_1\phi_1(n)\nu_n,$$

provided that (*a*) $\psi_n \to \infty$, (*b*) $\Delta\psi_n \to 0$, (*c*) $\Delta\psi_n/\nu_n \to \infty$ steadily,

and (*d*) $\Delta\psi_n/\nu_n < \phi_1(n)$.

We must now show that it is possible to choose ψ_n so as to satisfy these conditions. Since $\Sigma\mu_n$ is divergent, it follows, by a well known result due to Abel, that $\Sigma\nu_n$ is divergent. Hence (*c*) includes (*a*). Again, since $\nu_n \to 0$, $\phi_1(n)$ can be chosen so that $\nu_n\phi_1(n) \to 0$, for example, by choosing for $\phi_1(n)$ the minimum value of $\phi(m)$ for $m \geqslant n$, or the minimum value of $\nu_m^{-\frac{1}{2}}$ for $m \geqslant n$, whichever of these values is least. Then (*d*) includes (*b*), and we have only to choose ψ_n so as to satisfy (*c*) and (*d*). This may be done by choosing

$$\Delta\psi_n = \nu_n[\phi_1(n)]^{\frac{1}{2}}.$$

The series $\Sigma a_n' = \Sigma K_1^{-1}a_n$ is now finitely oscillating and summable $R(1, \mu_n)$, while

$$|a_n'| < \phi_1(n)\nu_n < \phi(n)\mu_n/\lambda_n.$$

Also the condition that $\lim_{n\to\infty} |s_n' e^{-\lambda_n x}| = 0$ for $x > 0$ is satisfied, since s_n' is finitely oscillating. The theorem is therefore established.

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* The notation $O[f(x)]$ for any function $\phi(x)$ such that $|\phi(x)| < K|f(x)|$ is due to Landau.

8. The conditions of Theorem (C) are satisfied when λ_n is any function of less order than $e^{\epsilon n}$ for all values of ϵ, which increases in a regular manner. When, however, $\lambda_n \geqslant e^{\rho n}$, the theorem breaks down altogether. I have proved, for example, that if $\lambda_n = e^{\rho n}$, $n^{-1}\log|a_n| \to 0$, and $\rho >$ a certain numerical constant, then the convergence of Σa_n is implied by the existence of the generalised Abel's limit. I do not, however, propose to discuss the proof (which is quite simple) here, for I believe that *no* explicit restriction on the a's is necessary when $\lambda_n \geqslant e^{\rho n}$, and I hope to consider this question in another paper.

9. *Applications.*

(i) Poisson attempted to prove Fourier's theorem by showing that, if $f(x)$ is a continuous function, then

$$\lim_{r\to 1}\left[a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx)\, r^n\right] = f(x),$$

where $a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx)$ is the Fourier's series corresponding to $f(x)$.* He supposed this result to involve the convergence of the Fourier series; *i.e.*, he assumed the converse of Abel's theorem. It is, however, possible, by means of Theorem (B), to justify this inference in the case when $f(x)$ satisfies Dirichlet's conditions. It follows immediately from Theorem (B) that if

$$\left|\int_0^{2\pi} \begin{matrix}\sin\\ \cos\end{matrix} n\theta f(\theta)\, d\theta\right| < K/n, \tag{13}$$

then the Fourier's series is convergent and has the sum $f(x)$. Hence, in order to prove Fourier's theorem with Dirichlet's conditions, it is sufficient to establish the inequality (13). This may be done without difficulty: for example, if $f(x)$ is monotonic the result follows immediately from the second mean value theorem of the differential calculus.†

(ii) M. Fejér and Mr. Hardy‡ have generalised as follows a well known result due to Frobenius:

If Σa_n is summable (C1) to sum s, then

$$\lim_{x\to 1} \Sigma a_n x^{n^k} = s.$$

Mr. Hardy shows also that this result does *not* necessarily hold when the index n^k is replaced by a^n, or, roughly speaking, any index of still

* *Journal de l'école polyt.*, cah. 19, 1823, p. 404. Poisson's proof of this result does not satisfy modern requirements of rigour. The first complete proof was given by Schwarz (*Math. Abh.*, Vol. II, pp. 144, 175).

† Cp. "O.S.," p. 308. Fatou (*loc. cit.*) has made the corresponding deduction from Tauber's theorem, viz., that the Fourier series converges if the upper and lower derivates of $f(x)$ for $0 \leqslant x \leqslant 2\pi$ are bounded.

‡ Fejér, *Math. Ann.*, Vol. 58, p. 51. Hardy, *Quarterly Journal*, Vol. XXXVIII, p. 269.

higher order. Theorem (C), combined with Tauber's theorem,* enables us to show that the result ceases to be true when n^k is replaced by *any* function of higher order than n^k for every value of k.

More precisely :

If Σa_n *is summable* $(C1)$ *but not convergent, and if* λ_n, μ_n *satisfy the conditions of Theorem* (C), *together with* $n\mu_n/\lambda_n \to \infty$, *then* $\lim_{n\to 1} \Sigma a_n e^{-x\lambda}$ *does not necessarily exist.*

For, since $n\mu_n/\lambda_n \to \infty$, it follows from Theorem (C) that there exists a series Σa_n, non-convergent but summable $(C1)$, such that

$$|na_n| < n\mu_n/\lambda_n, \quad \text{or} \quad |a_n| < \mu_n/\lambda_n.$$

If, now, $\lim \Sigma a_n e^{-x\lambda_n}$ exists, Theorem (B) requires that Σa_n should converge, which is untrue. Hence the theorem.

It is easy to prove generalisations, relating to summability $R(1, \mu_n)$, of both the positive and the negative result.

[*Added May 18th*, 1911.—

(iii) The following theorem is an immediate deduction from Theorem (B).

If $\lim \Sigma a_n e^{-xn}$ *exists*,† *and* $|n^{1-\alpha} a_n| < K$, *where* $\alpha > 0$, *then* $\Sigma n^{-\alpha} a_n$ *is convergent.*

For
$$\Sigma n^{-\alpha} a_n e^{-xn} = \int_0^\infty \Sigma a_n e^{-(x+t)n} t^{\alpha-1}\, dt.$$

It is not difficult to show that, with our assumptions, the right-hand side tends to a definite limit as $x \to 0$: the result then follows at once.

A similar theorem holds when $n^{-\alpha}$ is replaced by any function $\phi(n)$ which tends to zero and can be expressed in the form

$$\phi(n) = \int_0^\infty e^{-nt} \psi(t)\, dt,$$

where $\int_0^\infty e^{-xt} \psi(t)\, dt$ is absolutely convergent for $x > 0$. For example, a possible form of $\phi(n)$ is $n^{-\alpha} (\log n)^{\rho_1} \ldots (\log_k n)^{\rho_k}$.]

10. We have seen that if $\Sigma a_n x^n \to s$ and $a_n \to 0$, Σa_n is not necessarily convergent. There remains, however, the possibility that Σa_n is necessarily summable by one of Cesàro's means. I will add, in conclusion, a note on this point.

The most obvious types of series for which Abel's limit exists, and for

* We shall use, as a matter of fact, the wider Theorem (B) in the proof, but Tauber's theorem suffices at the expense of a little further complication.

† Or, more generally, if $|\Sigma a\ e^{-xn}| < Kx^{-\alpha+\eta}$, where $\eta > 0$.

which $|a_n| < Kn^p$, are summable, and, so far as I know, no one has constructed a non-summable series with the two former properties. It is, however, quite easy to do this. For example, $\lim_{x\to 1} \sum_2^\infty n^{-1}(ln)^\alpha \exp[\iota ln(l_2 n)^2]x^n$ exists, while $\Sigma n^{-1}(ln)^\alpha \exp[\iota ln(l_2 n)^2]$ is not summable by any one of Cesàro's means, unless $\alpha \leqslant 0$, in which case it is convergent.

The latter part of this statement is easily verified. To establish the existence of Abel's limit we employ the formula, easily established by means of the theory of residues,

$$\sum_2^\infty n^{-1}(ln)^\alpha \exp[\iota ln(l_2 n)^2]x^n = \int_{\frac{3}{2}}^\infty y^{-1}(ly)^\alpha \exp[\iota ly\,(l_2 y)^2]x^y\,dy$$
$$-\int_{\frac{3}{2}}^{\frac{3}{2}+\infty\iota} y^{-1}(ly)^\alpha \exp[\iota ly(l_2 y)^2]x^y\,\frac{dy\,e^{-2\pi\iota y}}{1-e^{-2\pi\iota y}}$$
$$+\int_{\frac{3}{2}}^{\frac{3}{2}-\infty\iota} y^{-1}(ly)^\alpha \exp[\iota ly(l_2 y)^2]x^y\,\frac{dy}{e^{2\pi\iota y}-1}. \qquad (14)$$

In the first integral the integration may be taken along a line lying in the first quadrant. It is then easy to show that the limit of the right hand side of (14), as $x \to 1$, is obtained by writing $x = 1$ in the subjects of integration.

It appears, therefore, that $\Sigma a_n x^n \to s$ and $|na_n| < K\,(\log n)^\alpha$ do not necessarily imply the summability of Σa_n. There is thus a very strong presumption that whatever function $\phi(n)$, tending to infinity, may be chosen, $\Sigma a_n x^n \to s$ and $|na_n| < \phi(n)$ do not necessarily imply the summability of Σa_n. But the discussion of the lacuna seems difficult, and I do not propose to consider the matter further at present.

[*Added May 18th.*—The work of §§ 1–6 may be modified so as to establish the following theorem.

The series Σa_n *is summable* $R(1, \mu_n)$ *if it is known to be finitely oscillating, if* $\lim_{\rho\to 0} \Sigma a_n e^{-\rho\lambda_n}$ *exists, and if the index* λ_n *satisfies the conditions of Theorem* (B). In particular, Σa_n *is summable* (*C*1) *if it is finitely oscillating and if Abel's limit exists.* This last result may also be generalised as follows :—*If Abel's limit exists, and* Σa_n *is "finite* (*Cr*)," *i.e., if Cesàro's r-th mean is finite, then* Σa_n *is summable* ($C\,r+1$).

In the case of Cesàro summation, we start from the formula

$$x^2 \Sigma \sigma_n e^{-xn} \to s,$$

where $\sigma_n = s_1 + s_2 + \ldots + s_n$. The proof then follows the lines of that of Theorem (B), σ_n and s_n playing the rôles of s_n and a_n.]

ON THE INTEGRATION OF FOURIER SERIES

By W. H. Young, Sc.D., F.R.S.

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1. In a companion paper * presented to the Society, I have discussed the circumstances under which one of the factors of the integrand of a definite integral may be replaced by a series, which is then multiplied term-by-term by the other factor of the integrand, and the new series so obtained integrated term-by-term. It was there pointed out that the series substituted for the factor in question need not have that factor for sum, and it need not, indeed, even converge. The more novel of the theorems obtained related, in fact, precisely to this case. Now, the series of Fourier can only be shown to converge, even in general, in a very restricted class of cases. On the other hand, what may conveniently be called *the integrated Fourier series*, obtained from the original series by replacing each term by its simplest indefinite integral, necessarily converges uniformly to the integral of the function associated with the original series. In accordance therefore with one of the theorems of the paper cited, we are at liberty to substitute for any function, which is summable in Lebesgue's manner, constituting one of the factors of the integrand of an integral, its corresponding Fourier series, whether this last converges or not, provided only that the remaining factor of the integrand is a function of bounded variation. This result is, however, the only one that follows immediately from the theorems of the paper cited. There are, none the less, a number of interesting and important cases, distinct from this one, in which the mode of procedure in question may be adopted. It is the main object of the present paper to set these forth in order. As far as I know, this has not been attempted in any existing text-book or memoir. In Hobson's Treatise the subject is not directly touched on, and earlier writers were necessarily ignorant of the considerations from which these results follow.

* W. H. Young, "On the Theory of the Application of Expansions to Definite Integrals" (1910), p. 463 *infra*.

Occasion is also taken to obtain a couple of simple results concerning what I have elsewhere called the semi-integrability above and below of a Fourier series, both *per se* and when multiplied by another function.

In general, it is of considerable importance in the theory to know whether an oscillating series is semi-integrable above and below. It enables us, for example, to assert with confidence that the upper and lower functions of the integrated series are integrals, a circumstance of which I have taken advantage in a previous paper * on "Trigonometrical Series" presented to this Society. In the case of Fourier series, however, we know that the integrated series, as already mentioned, converges uniformly, and, we may add, converges uniformly to an integral. Consequently the fact that a Fourier series is semi-integrable above and below is not in itself of the same importance. Similar remarks apply to the theorems concerning semi-integrability above and below when a Fourier series is multiplied by a function of the usual type, viz., a function of bounded variation. I have therefore not thought it necessary to linger on the present occasion over the properties in question. The other results stated in the paper are of use to the physicist and applied mathematician in so far as they may from time to time have occasion to make use of Fourier Series. They are as follows:—

The process of evaluating an integral referred to above is always allowable if—

(i) both factors are such that their squares are summable;

(ii) either of the factors is of bounded variation, while the other factor is summable;

(iii) the factor which is replaced by its Fourier series is summable, the other factor is bounded, and the series got by integrating the final series term-by-term between the desired limits converges;

(iv) the factor which is replaced by its Fourier series has a Harnack-Lebesgue integral, the other factor has bounded variation, and the series got by integrating the final series term-by-term converges.

Moreover, in the last two cases, if the series got by integrating the final series term-by-term does not converge, when summed in the ordinary way, it necessarily does so when summed in the Cesàro way, and the sum so obtained is the value of the integral.

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* W. H. Young, "On the Conditions that a Trigonometrical Series should have the Fourier Form" (1910), *supra*, p. 421.

The first of the above results is due to Fatou,* though stated by him in a slightly different manner. Of the results in (ii), one follows from the paper first cited; it is, however, much simpler to obtain both simultaneously by means of the theory of Fourier series, as is here done. These results are so important, and so easily obtained, that it is difficult to believe that they have not yet been stated; but I have not been able to find any such statement or any reference to such statement. With regard to (iii), it is practically certain that Lebesgue has definitely formulated it in his own mind, as only a single step is required to obtain it from his recently published work.† Of (iv), however, I have not seen any trace.

It should be added that I have extended these results to the case when one or both of the limits of integration are infinite, so far as this is possible without the introduction of additional irrelevant conditions.

One remark may be made in conclusion. In the early days of the history of Fourier series term-by-term integration was carried out without any *arrière pensée*, both in the case of Fourier series and when the series considered was that obtained by multiplying the Fourier series term-by-term by another function. A later school then arose and objected, with justice, to the uncritical nature of the investigations of their predecessors and the consequently inconclusive character of the results obtained. This earlier school of critics demanded that the series should be shewn to be uniformly convergent before being integrated term-by-term. Hence arose the almost interminable discussions with regard to the uniformity or the non-uniformity of the convergence of Fourier series. Finally, in the light of the most recent researches, it is seen that the considerations as to the uniformity of the convergence play a comparatively unimportant part in the theory of the integration of series, and that, in the case of Fourier series, the presence or absence of such uniformity has little or no bearing on the subject at all. Indeed, as is evident from the theorems stated above, even the question of the convergence of the series is frequently of secondary interest.

2. Fatou has shewn that if the a's and b's are the Fourier constants of $f(x)$, and A's and B's those of $g(x)$, then, provided only that the squares of f and g are summable,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,g(x)\,dx = \tfrac{1}{2}a_0 A_0 + \Sigma(a_n A_n + b_n B_n). \qquad (1)$$

* P. Fatou, "Séries trigonométriques et Séries de Taylor" (1905), *Acta Math.*, Bd. 30, p. 335.

† *Loc. cit.*, *infra*, p. 456.

Hence we immediately have the following theorem:—

THEOREM 1.—*If $f(x)$ and $g(x)$ are such that their squares are summable, the integral of their product between any finite limits may be evaluated by term-by-term integration of the series obtained by multiplying the Fourier series of either term-by-term by the other, that is*

$$\operatorname*{Lt}_{n=\infty} \int_c^z s_n(x)\, g(x)\, dx = \int_c^z f(x)\, g(x)\, dx,$$

*where $s_n(x)$ is the sum of the first n terms of the Fourier series of $f(x)$, provided, in the case when the length of the interval of integration exceeds 2π, the function to be replaced by its Fourier series is periodic.**

First let (c, z) be any interval inside the interval $(-\pi, \pi)$, and let us replace $g(x)$ by the function which agrees with it inside this interval and is zero outside it. The new function is, of course, like the old, one whose square is summable, but its Fourier constants now take a new form, in which the limits of integration are c to z. Thus, the above equation (1) becomes

$$\int_c^z f(x)\, g(x)\, dx = \tfrac{1}{2}a_0 \int_c^z g(x)\, dx + \Sigma \left(a_n \int_c^z g(x) \cos nx\, dx + b_n \int_c^z g(x) \sin nx\, dx \right). \tag{2}$$

This proves the required result for any limits of integration between $-\pi$ and π.

Next suppose $f(x)$ periodic, and $g(x)$ not to be so. Then the equation (1) plainly holds for the limits $(\pi, 3\pi)$, provided the A's and B's be supposed obtained by integrating between π and 3π. A similar remark applies for the integral $(3\pi, 5\pi)$, and so on. Hence, adding all such equations, we see that equation (1) holds when the limits of integration on the left-hand side are any positive or negative odd multiples of π, provided only the limits of integration in the A's and B's are correspondingly adjusted. Hence equation (2) also holds as it stands, whatever the limits of integration c and z may be, so long as they are finite.

3. Before proving the next theorem, we must first show that *the Fourier series of a function of bounded variation has the property of what*

* Throughout the present paper the word "periodic" is, for shortness, used to denote that the function to which the term is applied has the period 2π.

I have called bounded convergence *—in other words, that, where the convergence is not uniform, the non-uniformity has never infinite measure—that is, the peak and chasm functions are finite, and therefore bounded.

Let $f(x)$ be the function of bounded variation whose Fourier series is under consideration. Then the sum of the first $(2n+1)$ terms of its Fourier series is

$$\frac{1}{\pi}\int_0^{\frac{1}{2}\pi}[f(x+2z)+f(x-2z)]\frac{\sin mz}{\sin z}\,dz,$$

where m is the odd integer $(2n+1)$. It is clearly then sufficient to show that this expression, regarded as a function of the ensemble (x, m) has, when taken numerically, a finite upper bound. Further, we need only prove the statement for one of the two integrals into which the given integral naturally splits up—say, that in which $f(x+2z)$ occurs. Finally, we shall lose no generality if we suppose $f(x+2z)$ to be a positive monotone function of z, since the function of bounded variation $f(x+2z)$ may be expressed in the form $P(x+2z)-N(x+2z)$, where $P(u)$ and $N(u)$ are both positive monotone increasing functions, being the positive and negative variations of $f(u)$.

But if $f(x+2z)$ is a positive monotone increasing function of $2z$, the same is true of

$$zf(x+2z)/\sin z \quad \left(0 \leqslant z \leqslant \frac{\pi}{2}\right);$$

hence, by the Second Theorem of the Mean,

$$\int_0^{\frac{1}{2}\pi}\frac{zf(x+2z)}{\sin z}\,\frac{\sin mz}{z}\,dz = \frac{\pi}{2}f(x+\pi-0)\int_0^{k}\frac{\sin mz}{z}\,dz,$$

where k is some quantity between 0 and $\frac{1}{2}\pi$.

But the integral on the right-hand side lies numerically between 0 and π, and the factor multiplying it is a bounded function of x. Hence the required result follows.

4. We are now able to prove the following theorem :—

THEOREM 2.—*If one of the two functions $f(x)$ and $g(x)$ has bounded variation, while the other is summable, the integral of their product between any finite limits may be evaluated by term-by-term integration of*

* The limiting function is $\frac{1}{2}\{f(x+0)+f(x-0)\}$, and differs therefore from $f(x)$ only at a countable set of points. Hence, the limiting funct may, whenever it occurs under the sign of integration, be taken to be $f(x)$.

the series obtained by multiplying the Fourier series of either term-by-term by the other function, provided only that, if the length of the interval of integration exceeds 2π, the function whose Fourier series is employed is periodic.

To prove this we suppose the Fourier series of that of the two functions to be chosen which is a function of bounded variation. It then follows, by a known theorem * taken in conjunction with the result of § 3, that term-by-term integration between any finite limits of integration is allowable.

Denoting the function of bounded variation by $f(x)$, we then have, integrating between the limits $-\pi$ and π, the equation (1) of § 2. But this equation is clearly symmetrical with respect to $f(x)$, $g(x)$ and their respective Fourier constants, and holds, therefore, equally well whether it is $f(x)$ or $g(x)$, which is the function of bounded variation. Hence also the further reasoning of that article applies here equally whether it be $f(x)$ or $g(x)$, which is the function of bounded variation. Thus, both parts of our theorem are true.

5. In the last article we have not, however, used all the information at our disposal, and we can, accordingly, go somewhat farther than was possible in the first theorem. In the first place, the theorem quoted in my companion paper applies equally whether the limits of integration are finite or infinite, provided only, of course, that the function which is summable in that theorem is summable in the whole infinite interval, or, more accurately, that its absolute value has a Lebesgue integral in the whole of that interval.

With this understanding, the equation (2) of § 2 holds when either c or z is infinite, provided $g(x)$ is the function which is summable in the infinite interval in question, while $f(x)$ is the function of bounded variation, the boundedness of the variation referring to any and every finite interval.

In the second place, it is worth noting that the convergence of the integrated series is in accordance with the same theorem uniform. We

* For a proof of this theorem, which states that if $s_n(x)$ converges boundedly to its limit $f(x)$ as n increases, and $g(x)$ possesses a Lebesgue integral, $\int s_n(x)\, g(x)\, dx$ converges to $\int f(x)\, g(x)\, dx$, see my companion paper on "The Application of Expansions to Definite Integrals," § 4.

had no information enabling us to make this statement with regard to the corresponding series in Theorem 1.

Next suppose that $g(x)$ is a function of bounded variation in the whole infinite interval, and that throughout that interval it is summable; then $g(x)$ certainly has zero as unique limit when x increases indefinitely, for otherwise it could not possess an integral over the infinite interval. The function $f(x)$ is any summable periodic function, and we shall assume, in the first instance, that

$$\int_{-\pi}^{\pi} f(x)\,dx = 0.$$

This assumption makes $\int_c^z f(x)\,dx$ periodic, and ensures that the Fourier series of $f(x)$ is deficient of its first term, so that the integrated series having only periodic terms, its n-th partial summation, say $S_n(x)$, is periodic. Hence, since when z lies between $-\pi$ and π, the integrated Fourier series in question converges boundedly to $\int_c^z f(x)\,dx$, the same will be true wherever z lies in the infinite interval of integration.

We are now able to apply Theorem 6 of the companion paper, the second of the sufficient conditions for integration over an infinite interval having been shewn to be satisfied. This proves that for such a function $f(x)$ the integral of $f(x)\,g(x)$ over the infinite interval may be evaluated by term-by-term integration in the manner desired.

If the condition imposed in the first instance on $f(x)$ is not satisfied, it is satisfied by $h(x)$, where

$$h(x) = f(x) - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = f(x) - \tfrac{1}{2}a_0.$$

Hence, since the n-th partial summation of the Fourier series of $h(x)$ is $s_n(x) - \frac{1}{2}a_0$, we have, by what has just been proved,

$$\operatorname*{Lt}_{n=\infty} \int_c^{\infty} [s_n(x) - \tfrac{1}{2}a_0]\,g(x)\,dx = \int_c^{\infty} [f(x) - \tfrac{1}{2}a_0]\,g(x)\,dx,$$

whence, since $g(x)$ has, by hypothesis, an integral over the infinite interval,

$$\operatorname*{Lt}_{n=\infty} \int_c^{\infty} s_n(x)\,g(x)\,dx = \int_c^{\infty} f(x)\,g(x)\,dx,$$

which proves that *Theorem* 2 *holds for an infinite interval of integration.*

6. In the next two theorems the function whose Fourier series is to be used is no longer a function of bounded variation. We therefore naturally

use Cesàro's mode of summation, instead of summing directly. The Cesàro partial summation is the mean of the first n partial summations, that is,

$$s'_n(x) = \tfrac{1}{2}a_0 + (a_1 \cos x + b_1 \sin x)\left(1-\frac{1}{n}\right) + \ldots$$
$$+ [a_{n-1}\cos(n-1)x + b_{n-1}\sin(n-1)x]\left(1-\frac{n-1}{n}\right)$$
$$= \frac{1}{2n\pi}\int_{-\pi}^{\pi} f(y)\left\{\frac{\sin \tfrac{1}{2}n(x-y)}{\sin \tfrac{1}{2}(x-y)}\right\}^2 dy,$$

the latter expression being Fejér's integral, and it converges to $f(x)$ always except at a set of content zero, so that, for our purposes, we may work with $f(x)$ as the limiting function. Moreover, when $f(x)$ is bounded, it is easily seen that the convergence is bounded, for, if $f(x)$ is numerically less than K, Fejér's integral is less than K times what it becomes when we put $f(x) = 1$, in which case the Fourier series reduces to its first term, which is unity, so that the Fejér integral has also the value unity.

7. Corresponding exactly to Theorem 2, we have, therefore, using the Cesàro method of summation, the following theorem :—

THEOREM 3.—*If one of the two functions $f(x)$ and $g(x)$ is bounded, while the other is summable, the integral of their product between any finite limits may be evaluated by term-by-term integration of the series obtained by multiplying the Fourier series of either term-by-term by the other function, provided this integrated product series converges, and assuming, if the interval of integration exceed 2π in length, that the function whose Fourier series is employed is periodic.*

To prove this * let us take the Fourier series of that one of the two functions which is bounded, say $f(x)$. Then, as mentioned in the preceding article, the Cesàro partial summation

$$s'_n = \tfrac{1}{2}a_0 + (a_1 \cos x + b_1 \sin x)\left(1-\frac{1}{n}\right) + \ldots$$
$$+ [a_{n-1}\cos(n-1)x + b_{n-1}\sin(n-1)x]\left(1-\frac{n-1}{n}\right)$$

converges boundedly to a function, which, for our purposes, may be taken to be $f(x)$.

* The argument is that used by Lebesgue, *Les Intégrales Singulières*, pp. 107 *seq.*, so far as it is here applicable.

Hence, by the theorem used in proving Theorem 2,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,g(x)\,dx = \operatorname*{Lt}_{n=\infty} \frac{1}{\pi}\int_{-\pi}^{\pi} s_n'(x)\,g(x)\,dx$$

$$= \operatorname*{Lt}_{n=\infty} \left\{ \tfrac{1}{2}a_0A_0 + (a_1A_1+b_1B_1)\left(1-\frac{1}{n}\right) + \ldots \right.$$

$$\left. + (a_{n-1}A_{n-1}+b_{n+1}B_{n+1})\left(1-\frac{n-1}{n}\right)\right\},$$

the quantities A_i and B_i being the Fourier constants of $g(x)$, as the quantities a_i and b_i are the Fourier constants of $f(x)$. The expression on the extreme right of the equality (1) whose limit is to be taken, is the Cesàro partial summation of the series

$$\tfrac{1}{2}a_0A_0 + \sum_{n=1} (a_nA_n+b_nB_n),$$

so that, if this latter series converges, its sum will be the limit in question, and will accordingly be equal to

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,g(x)\,dx.$$

In other words, the equation (1) of § 2 holds in this case.

We have now only to repeat the argument of § 2, with the slight modification that we work in the first instance with the Cesàro partial summations instead of summing directly. Thus, the required theorem follows when the limits of integration lie in the first instance between $-\pi$ and π, and then, provided $f(x)$ is periodic for any finite interval of integration.

The symmetry of equation (1) of § 2 shews that, having been shewn to hold when $f(x)$ is bounded and $g(x)$ is summable, it holds equally when the rôles of these two functions are exchanged; the subsequent argument of § 2 then still holds, so that the theorem, having been shewn to be true in the former case, is equally true when $g(x)$ is bounded and $f(x)$ is summable. Thus, the theorem has been completely proved for every finite interval of integration.

COR.—*Even if the integrated product series does not converge directly, it will necessarily converge when summed in the Cesàro manner, and the sum so obtained is the value of the integral required.*

8. It is evident that we may extend the theorem of the preceding article to the case in which the interval of integration is infinite when $f(x)$

is the bounded function and $g(x)$ is the summable function. Then, as Theorem 2 of the companion paper holds equally whether the interval of integration is finite or infinite, the process carried out in the preceding article is still valid. Hence the extension is allowable.

9. We now come to the last of our theorems, which is concerned with the expression for the integral of the product of two functions. This is as follows :—

THEOREM 4.—*If one of the functions $f(x)$ and $g(x)$ is of bounded variation, while the other possesses a Harnack-Lebesgue integral, and if the series obtained by multiplying the Fourier series of either term-by-term by the other function converges when integrated between the chosen finite limits, then the integral of the product of the two functions between these limits is equal to the sum of the integrated series, provided only that, if the length of the interval of integration exceeds 2π, the function whose Fourier series is employed is periodic.*

It is clearly sufficient to prove the theorem when $f(x)$ is a function of bounded variation and $g(x)$ possesses a Harnack-Lebesgue integral. We then have, by the theory of Fejér's integral,

$$\tfrac{1}{2}[f(x+0)+f(x-0)] = \operatorname*{Lt}_{n=\infty} \frac{1}{2n\pi} \int_{-\pi}^{\pi} f(t) \left(\frac{\sin \tfrac{1}{2}n(t-x)}{\sin \tfrac{1}{2}(t-x)}\right)^2 dt$$

$$= \operatorname*{Lt}_{n=\infty} \int_{x-c}^{x+c} f(t) \left(\frac{\sin \tfrac{1}{2}n(t-x)}{\sin \tfrac{1}{2}(t-x)}\right)^2 dt.$$

Put $t = x+u$, and we get

$$\tfrac{1}{2}[f(x+0)+f(x-0)] = \operatorname*{Lt}_{n=\infty} \frac{1}{2n\pi} \int_{-c}^{c} f(u+x) \left(\frac{\sin \tfrac{1}{2}nu}{\sin \tfrac{1}{2}u}\right)^2 du$$

$$= \operatorname*{Lt}_{n=\infty} \frac{1}{2n\pi} \int_{0}^{c} [f(u+x)+f(x-u)] \left(\frac{\sin \tfrac{1}{2}nu}{\sin \tfrac{1}{2}u}\right)^2 du$$

$$= \operatorname*{Lt}_{n=\infty} F(x, n) = \operatorname*{Lt}_{n=\infty} \{F_1(x, n) - F_2(x, n)\},$$

where $F_1(x, n)$ and $F_2(x, n)$ are monotone functions of x. Also the left-hand side is equal to $f(x)$, except at a set of content zero. Hence, by a theorem in the Harnack integration of series,* whatever function $g(x)$ may be,

* See § 3 of the companion paper, p. 466 *infra*.

provided only it possess a Harnack-Lebesgue integral, we have, since the sequence is certainly bounded,

$$\int_{-\pi}^{\pi} f(x)\, g(x)\, dx = \operatorname*{Lt}_{n=\infty} \int_{-\pi}^{\pi} g(x)\, F(x, n)\, dx.$$

Retracing our steps, with the modifications required by the presence of the factor $g(x)$, we accordingly have

$$\int_{-\pi}^{\pi} f(x)\, g(x)\, dx = \operatorname*{Lt}_{n=\infty} \frac{1}{2n\pi} \int_{-\pi}^{\pi} dx\, g(x) \int_{-\pi}^{\pi} f(t) \left(\frac{\sin \frac{1}{2} n(t-x)}{\sin \frac{1}{2}(t-x)} \right)^2 dt.$$

From this point the argument is precisely the same as in the proof of the preceding theorem.

We have, of course, the same corollary as in the preceding article.

Cor.—*Even if the series obtained in the manner described does not converge in the ordinary manner, it will necessarily converge when summed in the Cesàro manner, and the sum so obtained is the value of the integral required.*

10. We have now to extend the theorem of the preceding article to the case of an infinite interval of integration.

When $f(x)$ is the function of bounded variation, the result of the preceding article remains true, since the argument used at the end of § 3 of the companion paper for an infinite interval of integration is again valid. Hence the theorem holds for an infinite interval of integration when $f(x)$ is a function of bounded variation.

Next let $g(x)$ be of bounded variation throughout the infinite interval, and let $f(x)$ possess a Harnack-Lebesgue integral in the same interval. When the limits c and z of integration lie in the closed interval $(-\pi, \pi)$, we know* that the integrated Fourier series of $f(x)$ is the Fourier series of $\int_c^z f(x)\, dx$, so that the Cesàro partial summation $s_n(x)$ of that series converges boundedly (in fact uniformly)† to $\int_c^z f(x)\, dx$. Hence, as in § 2 of the present paper, if

$$\int_{-\pi}^{\pi} f(x)\, dx = 0,$$

these facts remain true wherever the points c and z may lie in the infinite

* "On the Conditions that a Trigonometrical Series should have the Fourier Form," § 3, p. 425, *supra*.

† H. Lebesgue, *Sur les Intégrales singulières*, p. 89.

integral of integration. In this case, *provided $g(x)$ has zero for unique limit when x moves off to infinity*, we may apply Theorem 6 of the companion paper, the second of the sets of sufficient conditions there given being satisfied. This proves the theorem for an infinite interval of integration in this case.

If, however, $\frac{1}{2}a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$ is different from zero, the above shews that the theorem is true when $f(x)$ is replaced by $f(x)-\frac{1}{2}a_0$. That is,

$$\operatorname*{Lt}_{n=\infty}\int_c^{\infty}[s_n(x)-\tfrac{1}{2}a_0]\,g(x)\,dx = \int_c^{\infty}[f(x)-\tfrac{1}{2}a_0]\,g(x)\,dx,$$

from which, *provided $g(x)$ possesses an integral* over the infinite interval*, the required result follows, viz.,

$$\operatorname*{Lt}_{n=}\int_c^{\infty} s_n(x)\,g(x)\,dx = \int_c^{\infty} f(x)\,g(x)\,dx.$$

Here $s_n(x)$ is the partial summation, or the Cesàro partial summation, according as the Fourier series of $f(x)$, after term-by-term multiplication by $g(x)$ and integration, converges directly, or only when summed in the Cesàro manner.

11. In the previous part of the paper we have been concerned more with the function to which a given Fourier series belongs than with the Fourier series itself. If we fix our attention on the latter, in the case in which it does not converge in general or at all, it will have an upper function and a lower function, which may or may not be bounded, or even summable. If, however, we include for the moment among the summable functions those functions which give, when the Lebesgue process of dealing with unbounded integrands is applied, infinite values with definite signs, we can enunciate the following theorem:—

THEOREM 5.—*The upper and lower functions of every Fourier series are always summable, and the Fourier series itself is always semi-integrable, both above and below in the extended sense.*

In fact, the application of Poisson's method of dealing with Fourier series leads, with the use of Abel's Theorem, to the result that, wherever the function to which the Fourier series belongs is the differential coefficient of its integral, it lies between the upper and lower functions of the

* This will be an ordinary improper integral, since a function of bounded variation is, in every finite interval, integrable in the Riemann manner.

Fourier series. This is therefore true, except at a set of content zero. But the function to which the Fourier series belongs has necessarily a Lebesgue integral, for otherwise the Fourier series would not exist. Hence the upper and lower functions have Lebesgue integrals, in the sense explained above, and the integral of the function to which the series belongs is intermediate to these two integrals in value. Hence, bearing in mind that the integrated Fourier series converges to the integral of the function to which the series belongs, the result enunciated follows.

COR.—*If the upper and lower functions of a trigonometrical series be both bounded, the series is necessarily semi-integrable both above and below.*

This follows by the theorem* that such a series is necessarily a Fourier series; the result may also be extended to the case when the points at which the upper and lower functions of the series are unbounded are countable.

12. The result of the preceding article has been obtained by means of the known fact that a Fourier series, when integrated, converges to the integral of the function to which the Fourier series belongs. Similarly, by using Theorem 2 of the present paper, we can prove the following:—

THEOREM 6.—*The series obtained by multiplying the Fourier series of a function $f(x)$ term-by-term by a function $g(x)$ of bounded variation has its upper and lower functions always summable, and is always semi-integrable both above and below in the extended sense.*

First suppose $g(x)$ to be positive, and let $l(x)$ and $u(x)$ denote the lower and upper functions of the Fourier series. Then, by the argument of the preceding article, we have, except at most at a set of content zero,

$$l(x) \leqslant f(x) \leqslant u(x);$$

and therefore, multiplying by $g(x)$ and using Theorem 2, we have

$$\int l(x)\, g(x)\, dx \leqslant \operatorname*{Lt}_{n=\infty} \int s_n(x)\, g(x)\, dx \leqslant \int u(x) g(x) dx,$$

which proves the theorem in this case, and proves also that the integrals $\int l(x)\, g(x) dx$ and $\int u(x)\, g(x) dx$ exist, if we include respectively $-\infty$ and $+\infty$ as possible values of these integrals, for the limit in the middle necessarily exists and is finite, being equal to the integral of the product of a summable function by a function of bounded variation.

* "On the Conditions that a Trigonometrical Series should have the Fourier Form," § 6, p. 427, *supra*.

Next suppose $g(x)$ not to be positive, and let $g_1(x)$ be equal to $g(x)$ when $g(x)$ is positive, and be zero elsewhere, while $g_2 = -g(x)$ when $g(x)$ is negative and is zero elsewhere. Then $g_1(x)$ and $g_2(x)$ are functions of bounded variation, so that the inequalities (1) hold both for $g_1(x)$ and for $g_2(x)$. These give at once,

$$\int [l(x)\, g_1(x) - u(x)\, g_2(x)]\, dx = \operatorname*{Lt}_{n=\infty} \int s_n(x)\, g(x)\, dx$$
$$= \int [u(x)\, g_1(x) - l(x)\, g_2(x)]\, dx,$$

the first and last members of the inequality necessarily existing, at least to the extended sense above mentioned.

Now $l(x)\, g_1(x) - u(x)\, g_2(x) = l(x)\, g(x)$ when $g(x)$ is positive,

and $= u(x)\, g(x)$ when $g(x)$ is negative.

It is therefore the lower function of the succession $g(x)\, s_n(x)$. Similarly $u(x)\, g_1(x) - l(x)\, g_2(x)$ is the upper function. Hence the theorem follows, as enunciated.

COR.—*We have, of course, a corollary precisely similar to that of the preceding article.*

ON THE THEORY OF THE APPLICATION OF EXPANSIONS TO DEFINITE INTEGRALS

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1. In the application of expansions to definite integrals, either the whole integrand or one of its factors is replaced by an infinite series. The new infinite series, now constituted by the integrand, is then integrated term-by-term. The special case in which the integrand consists of a single function will not be discussed here; the problem of determining whether the substitution is in that case allowable is merely the general problem of the term-by-term integration of infinite series. In the more general case, the series which we substitute for one of the factors may converge everywhere to that factor as sum, or it may do so except at a set of content zero, or it may not have that function as sum, or even converge other than at exceptional points. We have an example of the last-named case when we substitute for one of the factors its series of Fourier; if the factor in question is a function of a general character, its Fourier series will not converge at all. On the other hand, the series got by integrating the Fourier series term-by-term always converges; it converges, in fact, uniformly to one of the integrals of the function associated with the original series. The example given by the application of the Fourier expansion suggests the general problem. Let the integrand of the integral to be considered be $f(x)\,g(x)$. If $f(x)$ can be expanded in a converging series, may we integrate this series when multiplied term-by-term by $g(x)$? If we cannot conveniently replace $f(x)$ by such a series, expand its integral $F(x)$ and differentiate the series so obtained term-by-term. Can we then integrate term-by-term the series got by multiplying term-by-term this last series by $g(x)$, and, if so, shall we in this way obtain $\int f(x)\,g(x)\,dx$? If $F(x)$ cannot be conveniently expanded, and its integral can, take the series representing this latter integral and differentiate it twice. Can the new series, when multiplied term-by-term by $g(x)$, be integrated term-by-term, and is the sum in this case $\int f(x)\,g(x)\,dx$?

This problem is of great practical importance, and its solution enables us to deal in a systematic manner with integrals whose treatment by ordinary methods involves special artifices.

In the case when $f(x)$ itself is capable of expansion the problem has already received partial solution, but, for completeness, I have given here, with indications of proof, the enunciations of the corresponding theorems, as well as the enunciations and detailed proofs of those that are new. I denote by $s_n(x)$ the n-th partial summation of what I call *the original series;* this is the series representing $f(x)$, when such a series exists, and otherwise is the series got by differentiating term-by-term a series giving the value of $\int f(x)dx$ or $F(x)$; or, more generally, is the series got by differentiating p times a series giving the p-th integral of $f(x)$. With this understanding, the process in question is always allowable as a means of evaluating the integral under the following circumstances:—

(i) If $s_n(x)$ is a monotone function of x, and the succession $s_n(x)$ is a sequence,* while $g(x)$ is any function possessing a Harnack-Lebesgue integral.

(ii) If $s_n(x)$ converges boundedly† as n increases, while $g(x)$ may be any function possessing a Lebesgue integral, proper or improper.

(iii) If $s_n(x)$ converges, not necessarily boundedly, and $\int [s_n(x)]^2 dx$ is a bounded function of (x, n), while $g(x)$ is any function whose square has a Lebesgue integral, proper or improper.

(iv) If the functions $|s_n(x)|$ form an integrable sequence, while $g(x)$ is any bounded function.

(v) If $\int s_n(x)dx$ converges to $\int f(x)dx$, and $\int |s_n(x)|dx$ is bounded, while $g(x)$ is any bounded function possessing only discontinuities of the first kind.

(vi) If $\int s_n(x)dx$ converges boundedly to $\int f(x)dx$, while $g(x)$ is any function of bounded variation.

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* For brevity we have not usually distinguished between a sequence and a succession which converges except at a set of content zero. For the same reason I have not specially called attention to the fact that the functions which occur in the theorems or in the processes may, under certain circumstances, only exist at a set complementary to a set of content zero.

† We say that a sequence converges boundedly if $s_n(x)$ is a bounded function of the ensemble (x, n), so that there are no points of non-uniform convergence with infinite measure; in other words, that the peak and chasm functions are finite. The *boundedness* must, of course, be without exception, even when non-convergence is allowed at a set of points of content zero, in accordance with the previous footnote.

(vii) If at the limits of integration $g(x) = 0$ or $\int s_n(x)dx$ converges to the integral $F(x)$, and the repeated integral $\int dx \int s_n(x)dx$ converges everywhere to the integral of $F(x)$, while $g(x)$ is any function which possesses a differential coefficient of bounded variation; and so on.

In all these cases we notice that the convergence of the new series when integrated is a bounded convergence, and it is under circumstances which are specified—uniform convergence. The theorems are also true, with, in certain cases, limitations which are given, when the range of integration is infinite.

I have purposely, in the present account, kept myself aloof from questions connected with the various methods in existence of attaching a conventional sum to a non-convergent series. To have done otherwise would have taken me too far afield. I hope none the less that the results obtained will be found interesting and useful, as well as systematic.

I have not thought it necessary to give more than a couple of examples; those selected will, perhaps, sufficiently serve the purpose of justifying the discussion here carried out. The first of these examples has been recently given* as illustrating the existence of one of "a number of comparatively simple cases that present themselves in practice, and do not come under any really general theorem." The second, almost equally simple, example has been specially constructed with the view of illustrating the final theorem of the present paper.

2. In the first four theorems which follow, it is immaterial whether we suppose the given function $f(x)$ to have itself been expanded in a series which converges everywhere except at most at a set of content zero, whose partial summation is $s_n(x)$, or whether we suppose $s_n(x)$ obtained by differentiating the corresponding partial summation $S_n(x)$ of the series which represents an indefinite integral of $f(x)$. In the latter case we must, of course, suppose that $S_n(x)$ is an integral, so that it is the integral of its differential coefficient $s_n(x)$. In fact, in all four theorems, $s_n(x)$ is seen to trace out an integrable sequence.

Calling its limiting function $s(x)$, we have therefore

$$\int s(x)dx = \operatorname*{Lt}_{n=\infty} \int s_n(x)dx = \operatorname*{Lt}_{n=\infty} S_n(x) = \int f(x)dx.$$

Hence $s(x)$ and $f(x)$ differ at most at a set of content zero, so that $f(x)$ may, under the sign of integration, replace the limit of $s_n(x)$ when n increases indefinitely.

* T. J. I'A. Bromwich, *Introduction to the Theory of Infinite Series* (1908), p. 451.

3. THEOREM 1.—*If the partial summations $s_n(x)$ are monotone ascending (or descending) functions of x and form a convergent sequence,* and $g(x)$ possesses a Harnack-Lebesgue integral, the sequence of Harnack-Lebesgue† integrals $\int_c^z s_n(x)\,g(x)\,dx$ converges to $\int_c^z f(x)\,g(x)\,dx$, and the convergence is uniform when the interval is finite. The convergence still holds when the interval is infinite, provided only the sequence is bounded at infinity.*‡

We may evidently assume $s_n(x)$ to be monotone decreasing, since in the alternative case we only have to change the variable from x to $-x$. The upper bound of the sequence $s_n(x)$ is then the upper bound of $s_n(c)$, and is therefore either the value of $s_n(c)$ for some integer n or the limiting value $s(c)$; in either case it is finite. Similarly the lower bound of the sequence is finite.

Again, adding a finite constant A, we get a bounded sequence of monotone decreasing positive functions. It follows then, by a theorem proved recently by myself in the *Messenger of Mathematics*,§ that the functions $g(x)\{A+s_n(x)\}$ form a sequence which is integrable term-by-term in the Harnack-Lebesgue manner. That is

$$\operatorname*{Lt}_{n=\infty} \int_c^z g(x)\{A+s_n(x)\}\,dx = \int_c^z g(x)\{A+f(x)\}\,dx,$$

whence also

$$\operatorname*{Lt}_{n=\infty} \int_c^z g(x)\,s_n(x)\,dx = \int_c^z g(x)\,f(x)\,dx,$$

where $f(x)$ has been written for the limiting function of $s_n(x)$ under the integral sign in accordance with § 2.

Thus the convergence of $\int_c^z s_n(x)\,g(x)\,dx$ to its limit $\int_c^z f(x)\,g(x)\,dx$ has been proved. In order to discuss the uniformity of the convergence, it will be convenient to re-prove the theorem quoted from the *Messenger*. Instead of employing Moore's conditions, I shall use the theorem‖ relating to the uniformity of approach of the defining sequence of integrals to the Harnack integral, defined as the limit of that sequence.

* See first footnote above. There may be oscillation, but it must be finite.

† That these are Harnack-Lebesgue integrals follows from the conditions of the problem. See E. W. Hobson, "The Second Mean-Value Theorem in the Integral Calculus," *Proc. London Math. Soc.*, Ser. 2, Vol. 7 (1908), p. 21.

‡ It is, of course, assumed in this case that $g(x)$ has a Harnack-Lebesgue integral in the whole infinite interval, that is, that the integral over a finite interval has a unique finite limit as the upper limit of integration moves off to infinity.

§ "On a Theorem in the Harnack Integration of Series" (1910), *Messenger of Mathematics*, pp. 101-106.

‖ See Hobson's *Theory of Functions of a Real Variable*, p. 383.

Assuming any positive quantity e, the theorem asserts that we can find a set of intervals d, surrounding the Harnack points, such that, if g_d denotes the function which is equal to g, except in the intervals d, where it is zero, then

$$\left| \int_c^z (g - g_d)\, dx \right| \leqslant e,$$

for all values of z in the range of integration. Hence, by the Second Theorem of the Mean,

$$\left| \int_c^z s_n (g - g_d)\, dx \right| \leqslant e s_n(c) \tag{1}$$

for all values of z. Here it has been supposed, for convenience, that the functions s_n are positive and monotone decreasing.

Similarly,

$$\left| \int_c^z f(x)\,(g - g_d)\, dx \right| \leqslant e f(c). \tag{2}$$

But, since the function g_d has no Harnack points, it has a Lebesgue integral; therefore, by Theorem 2 of the present paper, we can find m so that, for all values of n greater than m and all values of z,

$$\left| \int_c^z g_d (f - s_n)\, dx \right| \leqslant e. \tag{3}$$

From (1), (2), and (3) it follows that

$$\left| \int_c^z g(x)\,(f - s_n)\, dx \right| \leqslant eA,$$

where A is a finite quantity, independent of n or z. This proves that the left-hand side converges uniformly to zero, whence the required result at once follows for the special kind of sequence considered, and therefore, by the argument used above in the first instance, the theorem holds in all its generality for a finite interval of integration. To prove the convergence for an infinite interval, we have recourse to the general theory of change of order of limits, and make use of the Second Theorem of the Mean. In accordance with this theory, $\int_c^z s_n(x)\, g(x)\, dx$ will have its repeated limits, when z and n become infinite, equal, provided only that

$$\operatorname*{Lt}_{z=\infty} \operatorname*{Lt}_{n=\infty} \left| \int_z^\infty s_n(x)\, g(x)\, dx \right| = 0.$$

But, if B is the upper bound of the absolute value of $s_n(x)$ for all values of the ensemble (n, x), the modulus of the integral $\leqslant B \left| \int_z^{z'} g(x)\, dx \right|$, and

therefore, since $g(x)$ possesses a Harnack-Lebesgue integral in the whole infinite interval, the repeated limit in question is clearly zero.

Hence we may write

$$\int_c^\infty f(x)\,g(x)dx = \operatorname*{Lt}_{z=\infty}\int_c^z f(x)\,g(x)dx = \operatorname*{Lt}_{z=\infty}\operatorname*{Lt}_{n=\infty}\int_c^z s_n(x)\,g(x)dx$$

$$= \operatorname*{Lt}_{n=\infty}\operatorname*{Lt}_{z=\infty}\int_c^z s_n(x)\,g(x)dx = \operatorname*{Lt}_{n=\infty}\int_c^\infty s_n(x)\,g(x)\,dx,$$

which proves the theorem.

4. The following theorem is now well known, but it is rarely stated in its most general form.

Theorem 2.—*If $s_n(x)$ converges* boundedly as n increases and $g(x)$ is any function possessing a Lebesgue integral, proper or improper, the sequence of Lebesgue integrals $\int s_n(x)\,g(x)dx$ converges to $\int f(x)\,g(x)dx$, the limits of integration being finite or infinite. Moreover, the convergence is uniform throughout any finite interval.*

For, if U be any quantity greater than the upper bound of $|s_n(x)|$, for all values of n and x, $s_n(x)\,g(x)$ lies between $U|g(x)|$ and $-U|g(x)|$, which are both functions possessing Lebesgue integrals in the finite or infinite interval considered. Hence, by a known theorem,† since the functions $s_n(x)\,g(x)$ form a sequence, the unique limiting function of which may, for the purposes of integration (§ 2), be taken to be $f(x)\,g(x)$, this sequence is integrable so that

$$\int f(x)\,g(x)dx = \operatorname*{Lt}_{n=\infty}\int s_n(x)\,g(x)\,dx,$$

which proves the theorem as far as the convergence itself is concerned. The uniformity follows, for example, from the fact that,‡ if a sequence of functions, bounded below, is integrable, the sequence of integrals converges uniformly, for it is plain we shall obtain such a sequence if we add to each term of our sequence $U|g(x)|$.

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* See first footnote above, § 1.

† W. H. Young, "On Semi-integrals and Oscillating Successions of Functions" (1910), §§ 31 and 34, *supra*, pp. 286–325.

‡ Vitali, "Sulle Funzioni Integrali" (1905), *Atti di Torino*, Vol. lx, pp. 1021–1034. Cp. "On Semi-integrals and Oscillating Successions of Functions," § 25.

5. The next theorem is easily seen to follow from one given by myself elsewhere, and modified from Lebesgue.*

THEOREM 3.—*If* $s_n(x)$ *converges*† *and* $\int \{s_n(x)\}^2 dx$ *is bounded, then*

$$\operatorname*{Lt}_{n=\infty} \int s_n(x)\, g(x)\, dx = \int f(x)\, g(x)\, dx,$$

provided the square of $g(x)$ *possesses a Lebesgue integral, proper or improper; the limits of integration may here be finite or infinite. Moreover, the convergence of* $\int s_n(x)\, g(x)\, dx$ *to its limit is uniform throughout any finite interval.*

For, since $\int \{s_n(x)\}^2 dx$ is bounded, and the integrands, being positive and converging to $[s(x)]^2$, form a sequence which is semi-integrable below, we have

$$0 \leqslant \int [s(x)]^2 dx \leqslant \int [s_n(x)]^2 dx,$$

which proves that $[s(x)]^2$ possesses a Lebesgue integral, proper or improper. But, by hypothesis, $[g(x)]^2$ also possesses a Lebesgue integral, therefore

$$2\{[s(x)]^2 + [g(x)]^2\}$$

also possesses a Lebesgue integral. Hence, since this latter expression is equal to

$$\{s(x)+g(x)\}^2 + \{s(x)-g(x)\}^2,$$

both the squares in this expression have Lebesgue integrals, and therefore the same is true of half their difference, that is, $s(x)\, g(x)$.

Thus the conditions required in the theorem referred to in the footnote‡ are satisfied, so that the sequence of functions $s_n(x)\, g(x)$ is integrable. That is,

$$\int s(x)\, g(x)\, dx = \operatorname*{Lt}_{n=\infty} \int s_n(x)\, g(x)\, dx.$$

Putting $g(x) = 1$, it appears that the sequence of functions $s_n(x)$ is integrable, so that (§ 2) for purposes of integration we may replace $s(x)$ by $f(x)$, which proves the theorem as far as the convergence is concerned.

* Lebesgue's enunciation is as follows:—If $f(x)\, g(x)$ and $[g(x)]^2$ are summable, and if, further, $[f(x)-s_n(x)]^2$ is summable, and such that its integral is in the whole interval considered bounded, then the sequence $s_n(x)\, g(x)$, supposed to converge to $f(x)\, g(x)$, is integrable term-by-term (*Sur les Intégrales Singulières*, 1910, Fac. de Toulouse, 3e Série I, p. 50).

† See first footnote above, § 1.

‡ The theorem given in my paper on Semi-integrals, § 30, refers to successions of functions which do not necessarily converge anywhere. The special case of this theorem which we here require is the following:—If $f(x)\, g(x)$ and $[g(x)]^2$ are summable, and if, further, $[s_n(x)]^2$ is summable and such that its integral is bounded in the whole interval considered, then the succession $s_n(x)\, g(x)$, supposed to converge, except at a set of content zero, to $f(x)\, g(x)$, is absolutely integrable term-by-term.

That the convergence is uniform follows also from the theorem quoted, since the succession $s_n(x)\,g(x)$ is absolutely integrable. This theorem holds still when the limits of integration are not both finite, for in this case, by Schwarz's lemma,*

$$\left(\int_z^{z'} s_n g\,dx\right)^2 \leqslant \int_z^{z'} s_n^2\,dx \int_z^{z'} g^2\,dx \leqslant B\int_z^{z'} g^2\,dx, \tag{1}$$

where B is any positive quantity greater than the upper bound of $\int s_n^2\,dx$.

Now g^2 possesses an integral from c to ∞, so that when z moves off to infinity, z' remaining greater than z, the right-hand side has zero as unique limit, and therefore so has the left-hand side; therefore $s_n g$ has an integral from 0 to ∞.

Again
$$\left(\int_z^{z'} (s-s_n)g\,dx\right)^2 \leqslant \int_z^{z'} (s-s_n)^2\,dx \int_z^{z'} g^2\,dx$$
$$\leqslant \int_z^{z'} (s^2+s_n^2+2|ss_n|)\,dx \int_z^{z'} g^2\,dx. \tag{2}$$

But
$$\int_z^{z'} |2ss_n|\,dx \leqslant \int_z^{z'} s^2\,dx + \int_z^{z'} s_n^2\,dx;$$

therefore the left-hand side of (2) is
$$\leqslant \int_z^{z'} 2(s^2+s_n^2)\,dx \int_z^{z'} g^2\,dx.$$

Again, $s_n^2 \geqslant 0$, so that the sequence traced out by s_n^2 is semi-integrable below. Hence
$$\int_z^{z'} s^2\,dx \leqslant \operatorname*{Lt}_{n=\infty} \int_z^{z'} s_n^2\,dx \leqslant B.$$

Thus we get
$$\left(\int_z^{z'} (s-s_n)g\,dx\right)^2 \leqslant 4B\int_z^{z'} g^2\,dx.$$

As before, this shews that the left-hand side has zero as unique limit when z moves off to infinity, and proves that $(s-s_n)g$ is integrable from c to infinity, and therefore, since $s_n g$ has the same property, that sg is integrable from c to infinity.

Now
$$\int_c^{\infty} (s-s_n)g\,dx = \int_c^{z} (s-s_n)g\,dx + \int_z^{\infty} (s-s_n)g\,dx.$$

Here the first integral has, when z is fixed and n increases indefinitely, the unique limit zero, and the second integral is, by (3), such that, if n is fixed or not, provided only z moves off to infinity, this integral has the

* See, for instance, my paper on "A New Method in the Theory of Integration" (1910), *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 37.

unique limit zero. Thus, given any positive quantity e, we can find z and n so that

$$\int_c^\infty (s-s_n)g\,dx < e,$$

and this will be true for all greater values of n. Thus

$$\underset{n=\infty}{\text{Llt}} \int_c^\infty (s-s_n)g\,dx \leqslant e.$$

Since e is at our disposal, this shews that the limits on the left all coincide and have the value zero. Thus

$$\int_c^\infty s(x)\,g(x)dx = \underset{n=\infty}{\text{Lt}} \int_c^\infty s_n(x)\,g(x)dx.$$

6. The next theorem has, as far as I know, not been stated, but it is an immediate consequence of the theorem, given in my paper on Semi-integrals, that a sequence is integrable if two integrable sequences can be found, between which the given sequence everywhere lies.

THEOREM 4.—*If* $|s_n(x)|$ *traces out an integrable sequence and* $g(x)$ *is any bounded function, then*

$$\underset{n=\infty}{\text{Lt}} \int_c^z s_n(x)\,g(x)dx = \int_c^z f(x)\,g(x)dx,$$

the limits of integration being finite or infinite. Moreover, the convergence of $\int s_n(x)\,g(x)dx$ *to its limit is uniform throughout any finite interval.*

For, if U is any quantity greater than the upper bound of $|g(x)|$, $s_n(x)\,g(x)$ lies between $U|s_n(x)|$ and $-U|s_n(x)|$. Hence, by the theorem referred to above, the sequence traced out by $s_n(x)\,g(x)$ is integrable. In particular this is the case if $g(x) = 1$, so that, for purposes of integration, we may replace $s(x)$ by $f(x)$. Thus

$$\underset{n=\infty}{\text{Lt}} \int_c^z s_n(x)\,g(x)dx = \int_c^z f(x)\,g(x)dx,$$

which proves the theorem as far as the convergence is concerned.

That the convergence is uniform follows in precisely the same way as in Theorem 2, except that we now add the sequence $U|s_n(x)|$ term-by-term to the given sequence. Since the sequence so obtained, as well as the sequence added, is bounded below, each when integrated gives rise to a uniformly convergent sequence: hence the same is true of their difference, which proves the theorem.

7. The last theorem suggests a remark which it may be advisable to make. The theorems obtained state that the process in question can be carried out if, under the circumstances alleged, $g(x)$ is any member of a certain class of functions. It may, of course, also be allowable if $g(x)$ be a suitably chosen member of a different class. Thus in the last theorem it may be possible in a particular case to find an unbounded function $g(x)$ for which the result is true. An obvious example is that in which $|s_n(x)|$ is a monotone function of n, and $\underset{n=\infty}{\text{Lt}} \int |g(x)|\,|s_n(x)|\,dx$ exists or $\int |g(x)|\,|f(x)|\,dx$ exists.

This follows, for instance, from the theory of monotone sequences.

8. In the remaining theorems of the paper we may suppose that $f(x)$ cannot be conveniently expanded, and that instead we have recourse to the expansion of its integral or one of its repeated integrals. This may arise from the fact that $f(x)$ cannot be expanded in a series which satisfies any of the requirements of the Theorems 1–4 just given. In Theorems 5 and 6 we suppose $S_n(x)$ to be the partial summation of the expansion of $F(x)$, that is, $\int f(x)\,dx$, and write

$$s_n(x) = \frac{d}{dx}\, S_n(x).$$

In the most general case $S_n(x)$ must be an integral, so that $s_n(x)$ exists everywhere except at most at a set of content zero, where we may, if we please, attribute to it the value zero. With this understanding, we have the following theorem.

THEOREM 5.—*If* $\int |s_n(x)|\,dx$ *is bounded for all values of the ensemble* (x, n) *considered, and* $\int s_n(x)\,dx$ *or* $S_n(x)$ *converges to* $\int f(x)\,dx$ *or* $F(x)$, *while* $g(x)$ *is any bounded function possessing only discontinuities of the first kind, then*

$$\underset{n=\infty}{\text{Lt}} \int_c^z s_n(x)\, g(x)\, dx = \int_c^z f(x)\, g(x)\, dx.$$

This theorem remains true when one or both the limits of integration are infinite, provided only the original series when integrated represents a continuous function in the whole closed infinite interval.

Further, the convergence of $\int_c^z s_n(x)\, g(x)\, dx$ *to its limit is necessarily bounded, and it is uniform if the convergence of* $\int_c^z s_n(x)\, dx$ *to its limit is uniform.*

Since $g(x)$ has only discontinuities of the first kind, it has only a finite number of discontinuities at which the jump on one side at least is $\geqslant 1/p$, for at a limiting point of points of this kind the upper and lower limits on one side at least would differ by at least $1/p$, so that such a limiting point would be a discontinuity of the second kind. If we take the interval between two adjacent discontinuities $\geqslant 1/p$ and replace the value of $g(x)$ at the end-points by its limiting values on the sides towards the interval considered, so that these discontinuities are to be replaced by continuities, all the jumps in the closed interval will be less than $1/p$, so that, corresponding to each point we have an interval containing the point in which the oscillation of $g(x)$ is less than $1/p$. Hence, applying the Heine-Borel theorem, we can divide the interval into a finite number of parts in each of which the oscillation of the function is less than $1/p$.

Thus, when we have to do with a function $g(x)$ having only discontinuities of the first kind, we can always divide the continuum into a finite number of segments, such that the oscillation of $g(x)$ *inside* each segment is less than $1/p$, the oscillation inside the segment being supposed to take no account of the values of $g(x)$ at the end-points of the segment, but only of the limiting values there.

If we define a function $g_p(x)$ as equal to $g(x)$ at all the points of division, and equal to the upper bound* of $g(x)$ in each open segment at each point of that open segment, we shall have

$$| g(x) - g_p(x) | < 1/p,$$

so that the functions $g_p(x)$, as p increases, converge uniformly to $g(x)$.

Also, by the Second Theorem of the Mean or otherwise,

$$\left| \int_c^z \{g(x) - g_p(x)\} s_n(x)\, dx \right| < U/p, \tag{1}$$

provided $\int_c^z | s_n(x) |\, dx$ is, for all values of z and n, numerically less than U. Hence any limit of the left-hand side when n increases indefinitely is $\leqslant U/p$, so that, proceeding after taking the limit to let p increase indefinitely in such a way as to give a unique limit, we see that any repeated limit $\underset{p=\infty}{\text{Lt}}\ \underset{n=\infty}{\text{Lt}}$ of the left-hand side is zero. Thus

$$\underset{n=\infty}{\text{Lt}} \int_c^z g(x) s_n(x)\, dx = \underset{p=\infty}{\text{Lt}}\ \underset{n=\infty}{\text{Lt}} \int_c^z g_p(x)\, s_n(x)\, dx = \underset{p=\infty}{\text{Lt}} \int_c^z g_p(x) f(x)\, dx. \tag{2}$$

* Or any other value between the upper and lower bounds inclusive.

But the functions $g_p(x)$ form a bounded sequence, and $f(x)$ is a summable function; therefore, by a known theorem, already frequently used, the functions $g_p(x) f(x)$ form an integrable sequence, so that the last member of (2) is equal to

$$\int_c^z g(x) f(x)\, dx.$$

Hence also $$\operatorname*{Lt}_{n=\infty} \int_c^z g(x)\, s_n(x)\, dx = \int_c^z g(x) f(x)\, dx,$$

whatever sequence of values of n be taken so as to give a unique limit to the left-hand side of the inequality last given. Thus the limit in the preceding equation is unique, which proves the theorem.

That the convergence is always bounded and that it is uniform when the convergence of $\int s_n(x)dx$ to its limit is uniform, may be seen as follows. From (1) we deduce that this is the case for $g(x)$ if it is true for $g_p(x)$ for every value of p. But for fixed p, the result obviously holds, since g_p is constant in stretches.

If z is infinite the above reasoning still holds down to the inequality (1), in which we may put $z = \infty$, if we alter the sign $<$ to $\leqslant$, or which may be written

$$\operatorname*{Lt}_{z=\infty} \left| \int_c^z [g(x) - g_p(x)]\, s_n(x) \right| dx \leqslant U/p.$$

Hence, by the same reasoning as before, we get the equation corresponding to (2), which may be written

$$\operatorname*{Lt}_{p=\infty} \operatorname*{Lt}_{n=\infty} \left| \int_c^\infty [g(x) - g_p(x)]\, s_n(x) \right| dx = 0,$$

or $$\operatorname*{Lt}_{p=\infty} \operatorname*{Lt}_{n=\infty} \operatorname*{Lt}_{z=\infty} \left| \int_c^z [g(x) - g(p)]\, s_n(x) \right| dx = 0,$$

or $$\operatorname*{Lt}_{n=\infty} \int_c^\infty g(x)\, s_n(x)\, dx = \operatorname*{Lt}_{p=\infty} \operatorname*{Lt}_{n=\infty} \operatorname*{Lt}_{z=\infty} \int_c^z g_p(x)\, s_n(x)\, dx;$$

this we may write $$= \operatorname*{Lt}_{p=\infty} \operatorname*{Lt}_{n=\infty} \operatorname*{Lt}_{z=\infty} \left\{ \int_c^{z'_p} g_p(x)\, s_n(x)\, dx + \int_{z'_p}^z g_p(x)\, s_n(x)\, dx \right\},$$

where z'_p is the nearest of the points of division to the point infinity (with proper sign) considered. The first integral gives us no trouble, since it falls under the former case. In the latter integral g_p is constant, so that we may write the term

$$\operatorname*{Lt}_{p=\infty} g_p(x) \operatorname*{Lt}_{n=\infty} \operatorname*{Lt}_{z=\infty} \int_{z'_p}^z s_n(x)\, dx,$$

which is equal to

$$\operatorname*{Lt}_{p=\infty} g_p(x) \operatorname*{Lt}_{z=\infty} \operatorname*{Lt}_{n=\infty} \int_{z'_p}^{z} s_n(x)\,dx = \operatorname*{Lt}_{p=\infty} g_p(x) \operatorname*{Lt}_{z=\infty} \int_{z'_p}^{z} f(x)\,dx$$

$$= \operatorname*{Lt}_{p=\infty} \operatorname*{Lt}_{z=\infty} \int_{z'_p}^{z} g_p(x) f(x)\,dx,$$

if, and only if, the condition given in the enunciation holds, that is, if $\operatorname*{Lt}_{n=\infty} \int_{z'_p}^{z} s_n(x)\,dx$ represents a continuous function of z up to and including the point infinity. In this case, therefore, we have

$$\operatorname*{Lt}_{n=\infty} \int_{c}^{\infty} g(x)\, s_n(x)\,dx = \operatorname*{Lt}_{p=\infty} \int_{c}^{\infty} g_p(x) f(x)\,dx,$$

corresponding to equation (2) in the first case. The rest of the argument is unaltered.

9. In the preceding proof I have followed Lebesgue's argument in a kindred matter closely.* It would have been a little simplified if we had been content to state the theorem on the hypothesis that $g(x)$ is continuous, but the gain would not have been great. In the theorem about to be given the whole difficulty consists in the fact that we only require from $g(x)$ that it should be the most general kind of function of bounded variation. The proof of the theorem when $g(x)$ belongs to that particular subclass of the class of functions of bounded variation which are obtained by Lebesgue integration of other functions is immediate. The whole difficulty consists in the gradual extension of the theorem, first to continuous functions of bounded variation, and then to discontinuous functions of the same class. In the case which ordinarily arises $g(x)$ will have not only its first differential coefficient, but this differential coefficient will be finite, in which case $g(x)$ itself is an integral, viz., the integral of that differential coefficient; the importance of the theorem is then only matched by the simplicity of its proof. For theoretical purposes, however, the theorem is required in all its generality; I again follow Lebesgue in his treatment of a similar problem.

10. THEOREM 6.—*If* $\int s_n(x)\,dx$ *converges boundedly to* $\int f(x)\,dx$, *and* $g(x)$ *is any function of bounded variation in the interval considered, then*

$$\operatorname*{Lt}_{n=\infty} \int s_n(x)\, g(x)\,dx = \int f(x)\, g(x)\,dx.$$

* *Sur les Intégrales Singulières*, p. 59 *seq.*

Further the theorem remains true when one or both of the limits of integration is infinite, provid d only that either (i) $\underset{n=\infty}{\mathrm{Lt}} \int s_n(x)\,dx$ *is continuous at the point, or points, infinity, i.e., provided* $\int f(x)\,dx$ *exists, and is equal to the sum of the limits of the separate terms in the expansion of* $\int f(x)\,dx$, *or* (ii) $g(x)$ *converges to zero as* x *increases. Moreover, the convergence of* $\int s_n(x)\,g(x)\,dx$ *is necessarily bounded, and it is uniform if that of* $\int s_n(x)\,dx$ *is uniform.*

Note.—In the above theorem, when the interval of integration is infinite, we must make the obvious convention that a function which is of bounded variation in every finite interval is only of bounded variation in an infinite interval, if the limits (necessarily unique) of the positive and negative variations, are, as x increases or decreases indefinitely, finite.

If $g(x)$ be an integral we are able to use at once the theorem of integration by parts, and the required result follows by the reasoning below in an abbreviated form. Next, let $g(x)$ be a continuous function of bounded variation; we must then take a new variable, viz., t the arc of the curve $y = g(x)$.* Then t is a monotone continuous function of x, and therefore x is a monotone continuous function of t, and we may suppose it to be monotone increasing; y is also a function of t, so that we may write

$$x = x(t), \quad y = y(t).$$

Now the derivates of $y(t)$ lie between -1 and $+1$; therefore, by Lebesgue's theorem, dy/dt necessarily exists, except possibly for a set of values of t of content zero, and $y(t)$ is the integral of any one of its derivates. Thus

$$g[x(t)] = y(t) = \int \frac{dy}{dt}\,dt, \tag{1}$$

which shews that $g[(x, t)]$ is an integral with respect to t.

Also $$\int s_n(x)\,dx = S_n(x);$$

and therefore, since x is a monotone increasing function of t,

$$\int s_n[x(t)]\,\frac{dx}{dt}\,dt = S_n[x(t)], \tag{2}$$

so that the integral on the left has $F(x)$ as unique limit, when n increases

* Young's *Theory of Sets of Points*, p. 266.

indefinitely. Hence

$$\operatorname*{Lt}_{n=\infty} \int_c^z s_n(x)\, g(x)\, dx = \operatorname*{Lt}_{n=\infty} \int \{s_n[x(t)]\} \frac{dx}{dt}\, g[x(t)]\, dt$$

$$= \operatorname*{Lt}_{n=\infty} \left\{ g(z)\, S_n(z) - g(c)\, S_n(c) - \int S_n[x(t)] \frac{d}{dt} g[x(t)]\, dt \right\}, \tag{3}$$

since, by (1) and (2), both $g[x(t)]$ and $S_n[x(t)]$ are integrals with respect to t.

Since,* by hypothesis, the convergence of the integral series to $S(x)$ is bounded, $S_n(x)$ is a bounded function of the ensemble (x, n); and therefore $S_n[x(t)]$ is a bounded function of the ensemble (t, n). Therefore $S_n[x(t)] \frac{d}{dt} g(x, t)$ is numerically less than $A \left| \frac{d}{dt} g[x(t)] \right|$, where A is any quantity greater than the numerical values of the upper and lower bounds of $S_n[x(t)]$, for all values of the ensemble (t, n); that is, the integrand of the integral on the right of the preceding equation is numerically less than a summable function. Hence, by the theorem so frequently used, the sequence of functions of t,

$$S_n[x(t)] \frac{d}{dt} g[x(t)]$$

is integrable, so that the limits in the preceding equation all coincide, and we have

$$\operatorname*{Lt}_{n=\infty} \int_c^z s_n(x)\, g(x)\, dx = g(z) F(z) - g(c)\, F(c) - \int F[x(t)] \frac{d}{dt} g[x(t)]\, dt. \tag{4}$$

Now $$F(x) = \int f(x)\, dx,$$

so that, by substitution, $$F[x(t)] = \int f[x(t)] \frac{dx}{dt}\, dt. \tag{5}$$

Thus $F[x(t)]$, as well as $g[x(t)]$, is an integral with respect to t, so that, integrating by parts, we get from (3) the equation

$$\operatorname*{Lt}_{n=\infty} \int_c^z s_n(x) g(x) dx = \int g[x(t)] \frac{d}{dt} F[x(t)] dt = \int g[x(t)] f[x(t)] \frac{dx}{dt} dt,$$

since the differential coefficient of F, regarded as a function of t, is the integrand of the expression for F as an integral on the right-hand side of (4), except for a set of values of t of content zero.

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* We here repeat the argument used in proving Theorem 2, instead of merely quoting that theorem, with a view to the subsequent extension to an infinite interval given below.

Hence, by the formula for integration by substitution the required result follows,

$$\operatorname*{Lt}_{n=\infty} \int_c^z s_n(x)\, g(x)\, dx = \int_c^z g(x) f(x)\, dx. \tag{6}$$

Finally, let $g(x)$ be discontinuous; it has then only a countable set of discontinuities, and these are of the first kind. Let these, arranged in some order, be $p_1, p_2, \ldots$.

Since a function of bounded variation is the difference of two monotone increasing functions, we may assume that $g(x)$ is a monotone increasing function; if we prove the required formula (6) in this case, it will then follow, by subtraction, for the case when $g(x)$ is any function of bounded variation.

Let $J_r(x)$ denote the sum of those of the first r jumps which lie in the closed interval (c, x), where the jump at c is

$$g(c+0)-g(c),$$

and at x is

$$g(x)-g(x-0),$$

while at any intermediate point x' it is

$$g(x'+0)-g(x'-0),$$

always supposing these points belong to the first r discontinuities. Then $J_r(x)$ is a monotone function of x, constant in each of the half-open intervals, open on the right, into which the whole interval is divided by the first r discontinuities, and having at each of those discontinuities the same jumps on the left and right as $g(x)$ itself.

As r increases, J_r increases at each point x; therefore the monotone increasing sequence of functions $J_r(x)$ defines a unique limiting function $J(x)$, which, like the constituent functions of the sequence, is a monotone increasing function. $J(x)$ is evidently the sum of all the jumps of $g(x)$ in the closed interval (c, x), and is therefore not greater than $g(x)-g(c)$.

Hence $J(x+h)-J(x)$ is the sum of all the jumps in the closed interval $(x, x+h)$; therefore

$$g(x+0)-g(x) \leqslant J(x+h)-J(x) \leqslant g(x+h)-g(x),$$

whence, moving h up to zero,

$$J(x+0)-J(x) = g(x+0)-g(x), \tag{7}$$

which shews that $J(x)-g(x)$ is continuous on the right, and similarly on the left, and is therefore a continuous function, which, since both g and J are monotone ascending functions, is a function of bounded variation. By what has already been proved, therefore, the theorem is true when g is

replaced by $J-g$. It is therefore true, provided it holds when $g(x)$ is replaced by $J(x)$.

The formula certainly holds when $g(x)$ is replaced by $J_r(x)$, since $J_r(x)$ is constant in each of a finite number of segments making up the range of integration. Thus we have

$$\operatorname*{Lt}_{n=\infty} \int_c^z s_n(x)\, J_r(x)\, dx = \int_c^z f(x)\, J_r(x)\, dx. \tag{8}$$

Now $J(x)-J_r(x)$ is the sum of all the jumps after the r-th in the completely open interval (c, x), together with the left-hand jump at x, if x is neither a point of continuity, nor one of the first r discontinuities of $g(x)$. Hence $J(x)-J_r(x)$ is a monotone function of x, so that, by the Second Theorem of the Mean,

$$\left| \int_c^z s_n(x)\, [J(x)-J_r(x)]\, dx \right| \leqslant U\, |J(c)-J_r(c)|, \tag{9}$$

where U is the finite upper bound of the absolute value of $S_n(x)$. Hence the repeated limit, or limits $\operatorname*{Llt}_{r=\infty}\ \operatorname*{Llt}_{n=\infty}$ of the left-hand side are not greater than the limit of the right-hand side when r increases without limit. We have therefore

$$\operatorname*{Lt}_{r=\infty}\ \operatorname*{Llt}_{n=\infty} \int_c^z s_n(x)\, [J(x)-J_r(x)]\, dx = 0,$$

whence, by (8),

$$\operatorname*{Lt}_{n=\infty} \int_c^z s_n(x)\, J(x)\, dx = \operatorname*{Lt}_{r=\infty} \int_c^z g(x)\, J_r(x)\, dx. \tag{10}$$

Now $J_r(x)$ is bounded, say numerically less than A, and $g(x)$ is summable, therefore $g(x)\, J_r(x)$, lying between $Ag(x)$ and $-Ag(x)$, is, by the theorem already quoted, the general function of an integrable sequence. Hence, by (10),

$$\operatorname*{Lt}_{n=\infty} \int_c^z s_n(x)\, J(x)\, dx = \int_c^z g(x)\, J(x)\, dx,$$

which proves the required formula when $g(x)$ is replaced by $J(x)$, and therefore, as pointed out above, proves the theorem.

We have next to shew that the result continues to hold under the additional conditions stated, when the interval of integration is no longer finite. Let us first prove that condition (i) is sufficient. Evidently the proof holds as it stands when z is infinite, provided only we secure that equation (3) continues to hold. This requires

$$\int s_n[x(t)]\, \frac{d}{dt}\, g[x(t)]\, dt$$

to exist when the upper limit of integration is infinite. But this is the

case, since the integrand is less than $A\frac{d}{dt}g[x(t)]$, whose integral in any finite interval is the total variation of $g(x)$, and therefore has a unique finite limit as the interval stretches out to infinity. Since therefore the hypothesis (i) enables us to pass from (3) to (4), it is a sufficient condition for the truth of the theorem.

Next to shew that condition (ii) is sufficient. We may now suppose $g(x)$ expressed in the form $g_1(x)-g_2(x)$, where $g_1(x)$ and $g_2(x)$ are both monotone decreasing with zero as limit; it will therefore be sufficient to prove the theorem on the hypothesis that $g(x)$ itself is monotone decreasing and has the limit zero. Now the necessary and sufficient condition that the repeated limits

$$\operatorname*{Lt}_{z=\infty}\operatorname*{Lt}_{n=\infty}\int_c^z s_n(x)\,g(x)\,dx \quad \text{and} \quad \operatorname*{Lt}_{n=\infty}\operatorname*{Lt}_{z=\infty}\int_c^z s_n(x)\,g(x)\,dx,$$

should exist and be equal, is that

$$\operatorname*{Lt}_{z=\infty}\operatorname*{Lt}_{n=\infty}\int_z^\infty s_n(x)g(x)\,dx$$

should exist and be equal to zero.

By the Second Theorem of the Mean, this integral is, on the supposition stated, equal to

$$g(z)\int_z^{z'} s_n(x)\,dx.$$

But $\int_z^{z'} s_n(x)\,dx$ is a bounded function of the ensemble (n, z, z') by the conditions of the theorem in the whole infinite interval. Also $g(z)$ converges to zero as z increases. Hence the required result follows.

This proves the convergence of

$$\int_c^z s_n(x)\,g(x)\,dx \quad \text{to} \quad \int_c^z f(x)\,g(x)\,dx.$$

That the convergence is always bounded, and that it is uniform when that of $\int_c^z s_n(x)\,dx$ to its limit is uniform, follows, almost in the same manner as in the proof of the preceding theorem. From (9) it follows that the convergence will be uniform for $J(x)$, if it is so for $J_r(x)$ for all values of r. But for fixed r the convergence of $\int_c^z J_r(x)\,s_n(x)\,dx$ to its limit is of course uniform, if $\int_c^z s_n(x)\,dx$ converges uniformly, since J_r is constant in stretches. Hence $\int_c^z s_n(x)\,J(x)\,dx$ converges uniformly to $\int_c^z f(x)\,J(x)\,dx$.

Since $g(x)-J(x)$ is a continuous function of bounded variation, it only remains to prove the result in the case first treated (p. 476), and it will be true generally.

Now, by the same reasoning as before,

$$\int_c^z [s_n(x)-f(x)]\,g(x)\,dx = g(z)[S_n(z)-F(z)]-g(c)[S_n(c)-F(c)] - \int \{S_n[x(t)]-F[x(t)]\}\frac{dg}{dt}\,dt,$$

and when the convergence of $S_n(x)$ to $F(x)$ is uniform, we can confine our attention to such values of n that for all values of x

$$|S_n(x)-F(x)| < e,$$

so that the preceding equation gives

$$\int_c^z [s_n(x)-f(x)]\,g(x)\,dx < e\left[|g(z)|+|g(c)|+\int\left|\frac{dg}{dt}\right|dt\right] < eU,$$

where U is a finite constant. This shews that the left-hand side converges uniformly to zero, and therefore that the convergence of $\int_c^z s_n(x)\,g(x)\,dx$ in this case to its limit is also uniform. This completes the proof that, when $\int_c^z s_n(x)\,dx$ converges uniformly, so does $\int_c^z s_n(x)g(x)\,dx$. The same argument proves, without any condition, the boundedness of the convergence, e being now any quantity greater than a certain finite quantity. Thus the whole theorem is proved.

11. The preceding theorem is the first of a chain of similar theorems. The integral series, that is, the expansion of

$$F(x) = \int f(x)\,dx,$$

may be distinguished as the first integral series. Integrating a second time and expanding, say,

$$\int dx \int f(x)\,dx = \operatorname*{Lt}_{n=\infty} T_n(x),$$

the new series, whose partial summations are denoted by $T_n(x)$, may be called *the second integral series;* what we shall call *the original series* is then the series got by differentiating the second integral series term-by-term twice, so that

$$s_n(x) = T_n''(x).$$

Here, as already remarked, the equality may fail at a set of points of content zero without affecting our results, since the quantities involved only occur under the sign of integration. The functions $T'_n(x)$ are therefore supposed to be differentiable, except possibly at a set of points of content zero.

The series got by differentiating the second integral series term-by-term will not in general be what we have called the first integral series, nor does it even necessarily converge. If, however, the individual term or the individual partial summation $T'_n(x)$ is an integral, this series may be got by term-by-term integration of the original series; in particular this is the case if $s_n(x)$ is finite and summable. With this understanding we have the following theorem:—

THEOREM 7.—*If $T_n(x)$, the partial summation of the second integral series, is a bounded function of the ensemble (x, n), and $T'_n(x)$ is an integral, the series got by multiplying the original series term-by-term by a function $g(x)$ possessing a differential coefficient $g'(x)$ which is a function of bounded variation, when integrated term-by-term, converges to*

$$\int_c^z g(x) f(x)\, dx,$$

provided at the limits of integration $g(x)$ is zero, or $T'_n(x)$ converges to

$$\int f(x)\, dx.$$

For, by Theorem 6, denoting as before $\int_c^x f(x)dx$ by $F(x)$,

$$\operatorname*{Lt}_{n=\infty} \int_c^z T'_n(x)\, g'(x)\, dx = \int F(x) g'(x)\, dx. \tag{1}$$

But, since $g'(x)$ is a bounded function, $g(x)$ is an integral; also $F(x)$ and $T_n(x)$ are integrals; therefore, integrating by parts,

$$\int_c^z T'_n(x) g'(x)\, dx = \Big[T'_n(x)\, g(x) \Big]_c^z - \int_c^z T''_n(x)\, g(x)\, dx, \tag{2}$$

and

$$\int_c^z F(x)\, g'(x)\, dx = \Big[F(x)\, g(x) \Big]_c^z - \int_c^z f(x)\, g(x)\, dx. \tag{3}$$

From (1), (2) and (3) the theorem follows, under the conditions given in the enunciation, remembering that, under the sign of integration $T''_n(x)$ may be replaced by $s_n(x)$.

12. In precisely the same manner we can prove the following theorem :—

THEOREM 7′.—*If T_n, the partial summation of the third integral series, is a bounded function of the ensemble (x, n), and $T_n''(x)$ exists and is an integral, the series got by integrating term-by-term the original series multiplied term-by-term by a function $g(x)$ possessing a second differential coefficient $g''(x)$ which is a function of bounded variation, converges to $\int_c^z f(x)\,g(x)\,dx$, provided at the limits of integration either $g'(x) = 0$, or $T_n'(x)$ converges to $\int dx \int f(x)\,dx$, and either $g(x) = 0$, or $T_n''(x)$ converges to $\int f(x)\,dx$.*

The general theorem is at once evident :—

THEOREM $7^{(n-2)}$.—*If T_n, the partial summation of the r-th integral series, is a bounded function of the ensemble (x, n), and $T_n^{(r-1)}(x)$ exists and is an integral, the series got by integrating the original series term-by-term after multiplying term-by-term by a function $g(x)$, possessing an $(r-1)$-th differential coefficient which is a function of bounded variation, converges to $\int_c^z g(x)\,f(x)\,dx$, provided at the limits of integration either $g^{(k-1)} = 0$, or $T_n^{(r-k)}(x)$ converges to $T^{(r-k)}(x)$, for all values of k between 1 and $(r-1)$ inclusive.*

Here $T^{(r-k)}$ denotes the k-ple integral of $f(x)$.

It is unnecessary to write out the proofs of these theorems at length, as no new principle is involved.

13. We conclude the paper with two illustrative examples.

Ex. 1.*—Consider the integral $\int_0^1 x^p \log x\,(1+x)^{-2}\,dx$, where $p+1 > 0$.

Here let
$$f(x) = (1+x)^{-2} = 1-2x+3x^2-\ldots, \tag{1}$$
$$g(x) = x^p \log x, \tag{2}$$

and divide the integral into two parts, writing

$$\int_0^1 f(x)\,g(x)\,dx = \int_0^c f(x)\,g(x)\,dx + \int_c^1 f(x)\,g(x)\,dx.$$

* Bromwich, *loc. cit.*, p. 451.

If x lies between 0 and c, where c is less than unity, the series (1) converges boundedly, so that, by § 2, we may take it to be the original series. Also, by Theorem 2, if $g(x)$ is a summable function we may integrate term-by-term. This is the case in the first of the two integrals, which may accordingly be evaluated by term-by-term integration.

In the second integral the original series does not converge for $x = 1$, but the second integral series, viz., the expansion of $-\log(1+x)$ in a power series, converges boundedly (in fact uniformly). Hence, by Theorem 7, we may integrate term-by-term provided $g(x)$ has an integral for differential coefficient, which is certainly the case, since $x^p \log x$ possesses finite differential coefficients of every order, and provided secondly $g(x)$ is zero, or the first integral series converges, at the limits of integration. Now at the point $x = 1$, $g(x)$ is zero; and at the point $x = c$ the first integral series converges to $-(1+c)^{-1}$. Therefore the conditions of Theorem 7 are fulfilled, and we may evaluate the second integral, as well as the first by term-by-term integration, and therefore we may evaluate the given integral by term-by-term integration.

Ex. 2.—Let $$f(x) = (1+x)^{-1-q},$$

and $$g(x) = x^p (\log x)^q.$$

Here, as before, we write

$$\int_0^1 f(x)\, g(x)\, dx = \int_0^c f(x)\, g(x)\, dx + \int_c^1 f(x)\, g(x)\, dx. \qquad (1)$$

In treating the first of the two integrals on the right, we remark that $g(x)$ is summable; it has, in fact, for integral

$$\frac{x^{p+1}}{p+1}(\log x)^q - \frac{q x^{p+1}}{(p+1)^2}(\log x)^{q-1} + \frac{q(q-1)x^{p+1}}{(p+1)^3}(\log x)^{q-2} - \ldots$$

$$+ (-)^q \frac{x^{p+1}}{(p+1)^{q+1}}.$$

Also the series $$1 - (1+q)x + \frac{(1+q)(2+q)x^2}{2} - \ldots$$

converges boundedly to $(1+x)^{-1-q}$. Hence, by Theorem 2, term-by-term integration is allowable in dealing with the first integral.

In dealing with the second integral on the right of (1), we remark that the $(1+q)$-th integral series of $f(x)$, *i.e.*, the expansion of $\log(1+x)$ multi-

plied by a constant, converges boundedly, while $g(x)$ possesses differential coefficients of each order, so that the q-th is certainly an integral. Moreover all the differential coefficients up to the $(q-1)$-th inclusive are zero at $x=1$, while at $x=c$ the k-th integral series converges to $(1+x)^{-1-q+k}$, for all values of k from 1 to q inclusive. Therefore, by Theorem $7^{(q-1)}$, term-by-term integration is again allowable.

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